NDMI012: Combinatorics and Graph Theory 2

Lecture #7
Perfect graphs

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Remark: Recall that "maximal" means "inclusion-wise maximal," and "maximum" means "of maximum possible cardinality." This applies (for example) to cliques, stable sets, chains, and antichains.[1]

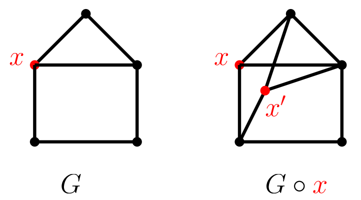
1 The Perfect Graph Theorem

Recall that a graph H is an induced subgraph of a graph G if $V(H) \subseteq V(G)$ and for all distinct $u, v \in V(H)$, we have that $uv \in E(H)$ if and only if $uv \in E(G)$.

Recall that a graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

In 1961, Berge conjectured that a graph is perfect if and only if its complement is perfect (this conjecture is known as the "Weak Perfect Graph Conjecture").[2] In 1972, Lovász proved the conjecture, which is now known as the Perfect Graph Theorem.

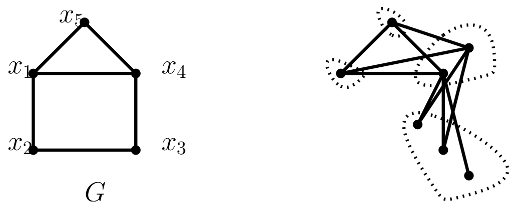
Duplicating a vertex x of a graph G produces a supergraph $G \circ x$ by adding to G a vertex x' and making it adjacent to all the neighbors of x in G, and to no other vertices of G (in particular, x and x' are nonadjacent in $G \circ x$). An example is shown below.



[1] Chains and antichains are defined in the section on Dilworth's theorem.

[2] Recall that for a graph G, the *complement* of G, denoted by $\overline{G}$, is the graph whose vertex set is $V(G)$, and in which any two distinct vertices are adjacent if and only if they are nonadjacent in G.

Vertex multiplication of a graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$ by a nonnegative integer vector $h = (h_1, \ldots, h_n)$ is the graph $G \circ h$ having h_i pairwise nonadjacent copies of x_i, such that copies of x_i and x_j are adjacent in $G \circ x$ if and only if $x_i x_j \in E(G)$. An example is shown below.



G

$h = (1, 0, 3, 2, 1)$ $G \circ h$

Recall that a *clique cover* of a graph G is a partition of $V(G)$ into cliques. The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the smallest size of a clique cover of G; a *minimum clique cover* of G is a clique cover of size precisely $\overline{\chi}(G)$. Clearly, $\overline{\chi}(G) = \chi(\overline{G})$ and $\alpha(G) \leq \overline{\chi}(G)$.

Initially, Berge defined two types of perfection, "χ-perfection" and "α-perfection."[3]

- A graph G is *χ-perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$.[4]
- A graph G is *α-perfect* if every induced subgraph H of G satisfies $\overline{\chi}(H) = \alpha(H)$.

Obviously, a graph is χ-perfect (i.e. perfect) if and only if its complement is α-perfect.

Lemma 1.1. *[Berge, 1961] Vertex multiplication preserves χ-perfection and α-perfection.*[5]

Proof.

Claim 1. Vertex duplication preserves χ-perfection.

Proof of Claim 1. Let G be a χ-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of a χ-perfect graph on fewer than $|V(G)|$ vertices is χ-perfect. Let $x \in V(G)$; we must show that $G \circ x$

[3]By the Perfect Graph Theorem, χ-perfection and α-perfection are equivalent. However, we have not proven this yet.

[4]In other words, χ-perfection is, by definition, the same as perfection.

[5]This means that for every graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$, and every nonnegative integer vector $h = (h_1, \ldots, h_n)$, we have the following:

- if G is χ-perfect, then so is $G \circ h$;
- if G is α-perfect, then so is $G \circ h$.

is χ-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\chi(G \circ x) = \omega(G \circ x)$, for the rest follows from the induction hypothesis. Clearly, we can extend an optimal coloring of G to a proper coloring of $G \circ x$, by giving x' the same color as x. So, $\chi(G \circ x) = \chi(G)$. Further, no clique contains both x and x', and it readily follows that $\omega(G \circ x) = \omega(G)$. Since G is χ-perfect, we have that $\chi(G) = \omega(G)$, and we now see that $\chi(G \circ x) = \chi(G) = \omega(G) = \omega(G \circ x)$. This proves Claim 1. ■

Claim 2. Vertex duplication preserves α-perfection.

Proof of Claim 2. Let G be an α-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of an α-perfect graph on fewer than $|V(G)|$ vertices is α-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is α-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\overline{\chi}(G \circ x) = \alpha(G \circ x)$, for the rest follows from the induction hypothesis.

Suppose first that x belongs to a maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G) + 1$. Since $\overline{\chi}(G) = \alpha(G)$ (because G is α-perfect), we can obtain a clique cover of size $\alpha(G) + 1$ by adding $\{x'\}$ as a one-vertex clique to some set of $\overline{\chi}(G)$ cliques covering G. This is enough because now we have that $\overline{\chi}(G) + 1 = \alpha(G) + 1 = \alpha(G \circ x) \leq \overline{\chi}(G \circ x) \leq \overline{\chi}(G) + 1$, and so $\overline{\chi}(G \circ x) = \alpha(G \circ x)$.

We may now assume that x does not belong to any maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G. Since $\overline{\chi}(G) = \alpha(G)$, Q intersects every maximum stable set of G. Since x belongs to no maximum stable set, $Q' = Q \setminus \{x\}$ also intersects every maximum stable set, and hence $\alpha(G \setminus Q') = \alpha(G) - 1$. Since G is α-perfect, $\overline{\chi}(G \setminus Q') = \alpha(G \setminus Q')$. To a set of $\alpha(G) - 1$ cliques covering $G \setminus Q'$, add the clique $Q' \cup \{x'\}$ to obtain a set of $\alpha(G) = \alpha(G \circ x)$ cliques covering $G \circ x$; we now have that $\overline{\chi}(G \circ x) = \alpha(G \circ x)$. This proves Claim 2. ■

Let G be a graph with vertex set $V(G) = \{x_1, \dots, x_n\}$, and let $h = (h_1, \dots, h_n)$ be a nonnegative integer vector. Let A be the set of vertices x_i for which $h_i > 0$. Clearly, if G is χ-perfect (resp. α-perfect), then $G[A]$ is also χ-perfect (resp. α-perfect). Now, $G \circ h$ can be obtained from $G[A]$ by a sequence of vertex duplications: if every h_i is 0 or 1 then $G \circ h = G[A]$, and otherwise, $G \circ h$ can be obtained from $G[A]$ by repeatedly duplicating vertices until there are h_i copies of each x_i. Since vertex duplication preserves χ-perfection and α-perfection (by Claims 1 and 2), an easy induction now guarantees that if G is χ-perfect (resp. α-perfect), then $G \circ h$ is also χ-perfect (resp. α-perfect). This completes the argument.

□

The Perfect Graph Theorem (Lovász, 1972)**.** *A graph is perfect if and only if its complement is perfect.*

Proof. Obviously, it is enough to prove that if a graph is perfect, then so is its complement. For this, it is in fact enough to prove that every α-perfect graph is χ-perfect, for then we will have the following sequence of implications for each graph G:

$$G \text{ is } (\chi\text{-})\text{perfect} \implies \overline{G} \text{ is } \alpha\text{-perfect} \implies \overline{G} \text{ is } (\chi\text{-})\text{perfect},$$

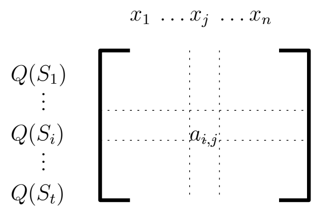
which is what we need.

Now, fix an α-perfect graph G, and assume inductively that all α-perfect graphs on fewer than $|V(G)|$ vertices are χ-perfect. We must show that G is χ-perfect. In view of the induction hypothesis, it suffices to show that $\chi(G) = \omega(G)$.[6]

Suppose first that G has a stable set S that intersects every maximum clique of G. Then by the minimality of G, $\chi(G \setminus S) = \omega(G \setminus S) = \omega(G) - 1$. But now $\chi(G) = \omega(G)$, since we can properly color $G \setminus S$ with $\omega(G) - 1$ colors, and then color all vertices of S with the same new color.

From now on, we assume that every stable set S of G misses (i.e. has an empty intersection with) some maximum clique $Q(S)$; our goal is to derive a contradiction. Set $V(G) = \{x_1, \ldots, x_n\}$, and let $\mathcal{S} = \{S_1, \ldots, S_t\}$ be the set of all maximal stable sets of G. For every vertex x_j, let h_j be the number of stable sets S in $\mathcal{S}$ such that $x_j \in Q(S)$. Set $h := (h_1, \ldots, h_n)$. By Lemma 1.1, $H := G \circ h$ is α-perfect, and so $\overline{\chi}(H) = \alpha(H)$.

Let $A = [a_{i,j}]_{t \times n}$ be a 0,1-matrix of the incidence relation between the set of $Q(S)$'s for $S \in \mathcal{S}$ and $V(G)$. So, $a_{i,j} = 1$ if and only if $x_j \in Q(S_i)$.



By construction, h_j is the number of 1's in column j of A, and $|V(H)|$ is the total number of 1's in A.

Since each row contributes $\omega(G)$ ones, we have $|V(H)| = \omega(G)|\mathcal{S}|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) \le \omega(G)$. Therefore $\overline{\chi}(H) \ge \frac{|V(H)|}{\omega(H)} \ge \frac{|V(H)|}{\omega(G)} = |\mathcal{S}|$.

Every stable set of H consists of copies of elements in some stable set of G; so, a maximum stable set of H consists of all copies of all vertices in some maximal stable set of G. Consequently,

$$\alpha(H) \;=\; \max_{T \in \mathcal{S}} \sum_{j : x_j \in T} h_j.$$

[6] Indeed, suppose that H is a proper induced subgraph of G. Then H is an α-perfect graph on fewer than $|V(G)|$ vertices, and consequently, H is χ-perfect. So, $\chi(H) = \omega(H)$. Thus, to show that G is χ-perfect, it suffices to show that $\chi(G) = \omega(G)$.

The sum above counts the 1's in A that appear in the columns indexed by T. If we count these 1's by rows, we get

$$\alpha(H) \quad = \quad \max_{T \in \mathcal{S}} \sum_{S \in \mathcal{S}} |T \cap Q(S)|.$$

Since T is a stable set, it has at most one vertex in each chosen clique $Q(S)$. Also, $T \cap Q(T) = \emptyset$. So, $|T \cap Q(S)| \leq 1$ for every $S \in \mathcal{S}$, and $|T \cap Q(T)| = 0$. It follows that $\alpha(H) \leq |\mathcal{S}| - 1$. Therefore $\alpha(H) < \overline{\chi}(H)$, contrary to the fact that H is α-perfect. □

2 Dilworth's theorem and comparability graphs

Recall that a *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.[7] A *partially ordered set* (or *poset*) is an ordered pair $(X, \preceq)$ such that X is a set and $\preceq$ is a partial order on X. A *maximal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $x \preceq y$.[8] Similarly, a *minimal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $y \preceq x$.[9] We say that $x, y \in X$ are *comparable* with respect to $\preceq$ if either $x \preceq y$ or $y \preceq x$; two elements of X are *incomparable* with respect to $\preceq$ if they are not comparable with respect to $\preceq$. A *chain* in $(X, \preceq)$ is a set $C \subseteq X$ such that any two elements of C are comparable with respect to $\preceq$. A *chain decomposition* of $(X, \preceq)$ is a partition of X into chains of $(X, \preceq)$. An *antichain* in $(X, \preceq)$ is a set $A \subseteq X$ such that no two elements of A are comparable with respect to $\preceq$.

Dilworth's theorem. *In any finite partially ordered set $(X, \preceq)$,[10] the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.*[11]

Proof. Let $(X, \preceq)$ be a finite partially ordered set, and assume inductively that the theorem is true for smaller partially ordered sets.[12] We may assume

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[7]A binary relation $\preceq$ on a set X is

- *reflexive* if for all $x \in X$, we have $x \preceq x$;
- *antisymmetric* if for all $x, y \in X$, if $x \preceq y$ and $y \preceq x$, then $x = y$;
- *transitive* if for all $x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

[8]A partially ordered set may or may not contain a maximal element. Furthermore, if a partially ordered set does contain a maximal element, then this maximal element may or may not be unique.

[9]A partially ordered set may or may not contain a minimal element. Furthermore, if a partially ordered set does contain a minimal element, then this minimal element may or may not be unique.

[10]Here, "finite" simply means that X is finite.

[11]The *size* of a chain decomposition is the number of chains in it.

[12]So, we are assuming that for all finite partially ordered sets $(X', \preceq')$ such that $|X'| < |X|$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X', \preceq')$.

that $X \neq \emptyset$, for otherwise, the result is immediate. First, it is clear that if $(X, \preceq)$ has an antichain of size k, then no chain decomposition of $(X, \preceq)$ is of size smaller than k (this is because no chain can contain two elements of an antichain).[13] It remains to exhibit an antichain of $(X, \preceq)$ and a chain decomposition of $(X, \preceq)$ of the same size.[14]

Since $(X, \preceq)$ is a nonempty, finite partial order, we see that $(X, \preceq)$ has a maximal element, say x_0. Set $X_0 := X \setminus \{x_0\}$, and let A_0 be a maximum antichain in $(X_0, \preceq)$;[15] set $k := |A_0|$. By the induction hypothesis, $(X_0, \preceq)$ has a chain decomposition of size k, say $\{C_1, \dots, C_k\}$.

> **Claim 1.** Any antichain of size k in $(X_0, \preceq)$ intersects each of $C_1, \dots, C_k$ in exactly one element.

Proof of Claim 1. Let B be a antichain of size k in $(X_0, \preceq)$. Since B is a antichain and $C_1, \dots, C_k$ are chains in $(X_0, \preceq)$, we see that B intersects each of $C_1, \dots, C_k$ in at most one element. But since $B \subseteq C_1 \cup \dots \cup C_k$, and since $|B| = k$, we see that B intersects each of $C_1, \dots, C_k$ in exactly one element. This proves Claim 1. ■

Now, for all $i \in \{1, \dots, k\}$, let C_i' be the set of all elements of C_i that belong to some antichain of $(X_0, \preceq)$ of size k; then $C_i' \neq \emptyset$ (because, by Claim 1, $C_i \cap A_0 \neq \emptyset$), and we deduce that C_i' has a unique maximal element, call it x_i.[16]

> **Claim 2.** $\{x_1, \dots, x_k\}$ is an antichain in $(X_0, \preceq)$.[17]

Proof of Claim 2. We may assume that $k \geq 2$, for otherwise, this is immediate. By symmetry, it suffices to show that x_1 and x_2 are incomparable. Let A_1 be an antichain of size k in $(X_0, \preceq)$ such that $x_1 \in A_1 \cap C_1$.[18] By Claim 1, $|A_1 \cap C_2| = 1$; set $A_1 \cap C_2 = \{x_2'\}$. Then $x_2' \in C_2'$, and so (since x_2 is a maximal element of the chain C_2') we have that $x_2' \preceq x_2$.[19] Now, if $x_2 \preceq x_1$, then by the transitivity of $\preceq$, we have that $x_2' \preceq x_1$, which is impossible since x_1 and x_2' are distinct elements of the antichain A_1.[20] So, $x_2 \not\preceq x_1$. An analogous argument establishes that $x_1 \not\preceq x_2$. Thus, x_1 and x_2 are incomparable. This proves Claim 2. ■

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[13] Thus, the maximum size of an antichain in $(X, \preceq)$ is no greater than the minimum size of a chain decomposition of $(X, \preceq)$.

[14] This will imply that the maximum size of an antichain in $(X, \preceq)$ is no smaller than the minimum size of a chain decomposition of $(X, \preceq)$.

[15] That is: A_0 is an antichain in $(X_0, \preceq)$ of largest possible cardinality.

[16] Since $C_i' \subseteq C_i$, and C_i is a chain, we know that C_i' is a chain. Furthermore, since X_0 is finite, C_i' is finite, and we already saw that C_i' is nonempty. So, C_i' is a nonempty, finite chain in $(X_0, \preceq)$, and it follows that it has a unique maximal element.

[17] Obviously, this means that $\{x_1, \dots, x_k\}$ is an antichain in $(X, \preceq)$ as well.

[18] Such an A_1 exists because $x_1 \in C_1'$.

[19] Since C_2' is a chain, we know that x_2', x_2 are comparable. Since x_2 is maximal in C_2, we have that $x_2' \preceq x_2$.

[20] The fact that $x_1 \neq x_2'$ follows from the fact that $x_1 \in C_1$, $x_2' \in C_2$, and $C_1 \cap C_2 = \emptyset$.

Suppose first that $\{x_0, x_1, \dots, x_k\}$ is an antichain in $(X, \preceq)$. Then this antichain is of size $k+1$, and $\{C_1, \dots, C_k, \{x_0\}\}$ is a chain decomposition of $(X, \preceq)$ of size $k+1$, and we are done. So, we may assume that $\{x_0, x_1, \dots, x_k\}$ is not an antichain in $(X, \preceq)$. By Claim 2, and by symmetry, we may assume that x_0 and x_1 are comparable; since x_0 is a maximal element of $(X, \preceq)$, we see that $x_1 \preceq x_0$. Now, set $D_1 := \{x_0\} \cup \{x \in C_1 \mid x \preceq x_1\}$; since C_1 is a chain, and $x_1 \preceq x_0$, the transitivity of $\preceq$ guarantees that D_1 is a chain in $(X, \preceq)$. Further, by Claim 1, and by the choice of x_1, we know that $(X \setminus D_1, \preceq)$ does not have an antichain of size k. Since $\{x_2, \dots, x_k\}$ is an antichain of size $k-1$ in $(X \setminus D_1, \preceq)$, we deduce that the maximum size of an antichain in $(X \setminus D_1, \preceq)$ is $k-1$. Then by the induction hypothesis, $(X \setminus D_1, \preceq)$ has a chain decomposition of size $k-1$, say $\{E_1, \dots, E_{k-1}\}$. But now $\{D_1, E_1, \dots, E_{k-1}\}$ is a chain decomposition of size k in $(X, \preceq)$, and we are done. □

A *comparability graph* (or a *transitively orientable graph*) is a graph G such that there exists a partial order $\preceq$ on $V(G)$ such that for all distinct $x, y \in V(G)$, we have that $xy \in E(G)$ if and only if x and y are comparable with respect to $\preceq$.[21] Equivalently,[22] G is a comparability graph if there exists an orientation $\vec{G} = (V(G), A(G))$ of G such that for all $\vec{uv}, \vec{vw} \in A(G)$, we have that $\vec{uw} \in A(G)$.

Corollary 2.1. *Every comparability graph is perfect. The complement of any comparability graph is perfect.*

Proof. In view of the Perfect Graph Theorem, it suffices to show that the complement of any comparability graph is perfect. So, fix a comparability graph G, and assume inductively that for all comparability graphs G' on fewer than $|V(G)|$ vertices, the graph $\overline{G'}$ is perfect. We must show that $\overline{G}$ is perfect. Clearly, it suffices to show that $\chi(\overline{G}) = \omega(\overline{G})$, for the rest follows from the induction hypothesis.[23]

Let $\preceq$ be a partial order on $V(G)$ such that for all distinct $x, y \in V(G)$, we have that $xy \in E(G)$ if and only if x and y are comparable with respect to $\preceq$. Let A be a maximum antichain in $(V(G), \preceq)$, and let $(C_1, \dots, C_k)$ be a chain decomposition of minimum size in $(V(G), \preceq)$. By Dilworth's theorem, we have that $|A| = k$. Now, note that A is a stable set in G, and therefore a clique in $\overline{G}$; so, $\omega(\overline{G}) \geq |A| = k$. On the other hand, $C_1, \dots, C_k$ are cliques in G, and therefore stable sets in $\overline{G}$; thus, $\{C_1, \dots, C_k\}$ is a partition of $V(\overline{G})$ into stable sets, and it follows that $\chi(\overline{G}) \leq k$. So, $\chi(\overline{G}) \leq k \leq \omega(\overline{G})$. But

[21] Note that in a comparability graph, cliques correspond to chains, and stable sets correspond to antichains.

[22] Check that this is really equivalent!

[23] Indeed, suppose that H is a proper induced subgraph of $\overline{G}$. Then $\overline{H}$ is a comparability graph, and so by the induction hypothesis, $\overline{\overline{H}} = H$ is perfect. Thus, $\chi(H) = \omega(H)$. It only remains to show that $\chi(\overline{G}) = \omega(\overline{G})$.

obviously, $\chi(\overline{G}) \geq \omega(\overline{G})$, and we deduce that $\chi(\overline{G}) = \omega(\overline{G})$. This completes the argument. $\square$

3 Some further examples of perfect graphs

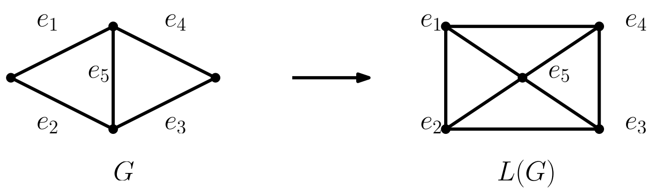
Lemma 3.1. *Every bipartite graph is perfect.*

Proof. Since all induced subgraphs of a bipartite graph are perfect, it suffices to show that every bipartite graph G satisfies $\chi(G) = \omega(G)$. But this is obvious: if G is an edgeless bipartite graph, then $\chi(G) = \omega(G) = 1$, and if G is a bipartite graph that has at least one edge, then $\chi(G) = \omega(G) = 2$. $\square$

Lemma 3.2. *The complement of any bipartite graph is perfect.*

Proof. This follows immediately from Lemma 3.1 and the Perfect Graph Theorem. $\square$

Given a graph G, the *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$, in which distinct $e, f \in E(G)$ are adjacent if and only if they share an endpoint in G. An example is shown below.



Lemma 3.3. *The line graph of any bipartite graph is perfect.*

Proof. Let G be a bipartite graph, and let $H = L(G)$. We must show that H is perfect. Consider any induced subgraph $H' = H[M]$ of H. So, we have that $M \subseteq V(H)$, and therefore, $M \subseteq E(G)$. Let G' be a subgraph of G with vertex set $V(G)$ and edge set M. Since H' is an induced subgraph of H, it follows that $H' = L(G')$, and consequently, $\chi(H') = \chi'(G')$. On the other hand, since G' is bipartite, Theorem 3.4 from Lecture Notes 5 guarantees that $\chi'(G') = \Delta(G')$. Clearly, $\Delta(G') \leq \omega(H')$, and so it follows that

$$\chi(H') \;=\; \chi'(G') \;=\; \Delta(G') \;\leq\; \omega(H').$$

Since we also know that $\chi(H') \geq \omega(H')$, we deduce that $\chi(H') = \omega(H')$. It follows that H is perfect. $\square$

Lemma 3.4. *The complement of the line graph of any bipartite graph is perfect.*

Proof. This follows immediately from Lemma 3.4 and the Perfect Graph Theorem. $\square$

4 The Strong Perfect Graph Theorem

A *hole* in a graph G is an induced cycle of length at least four.[24] An *antihole* in G is an induced subgraph H of G such that $\overline{H}$ is a hole in $\overline{G}$. An *odd hole* (resp. *odd antihole*) is a hole (resp. antihole) that has an odd number of vertices. Even holes and even antiholes are defined analogously. A graph is *Berge* if it contains no odd holes and no odd antiholes.

The Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour, Thomas, 2002)**.** *A graph is perfect if and only if it is Berge.*

Clearly, a graph is Berge if and only if its complement is Berge. So, the Strong Perfect Graph Theorem immediately implies the Perfect Graph Theorem.

One direction of the Strong Perfect Graph Theorem ("every perfect graph is Berge") is an easy exercise. Indeed, it is easy to check that for each integer $n \geq 2$, we have that

- $\omega(C_{2n+1}) = 2$ and $\chi(C_{2n+1}) = 3$;
- $\omega(\overline{C_{2n+1}}) = n$ and $\chi(\overline{C_{2n+1}}) = n + 1$.

So, odd holes and antiholes are imperfect, and therefore, no perfect graph contains an odd hole or an odd antihole. Thus, every perfect graph is Berge.

How about the other direction ("every Berge graph is perfect")? It relies on a "decomposition theorem" for Berge graphs, which, roughly, states that every Berge graph either is "basic" or admits a "decomposition." (The proof of this decomposition theorem is by far the most complicated part of the proof of the Strong Perfect Graph Theorem, and it is over 100 pages long.) The "basic" graphs are bipartite graphs and their complements, line graphs of bipartite graphs, complements of line graphs of bipartite graphs, and "double split" graphs (we omit the definition). All basic graphs are perfect: we proved this for the first four types of basic graphs, and the proof for double split graphs is easy. There are several "decompositions" (we omit the definitions), and it can be shown that no imperfect Berge graph of minimum possible size admits any of these decompositions. It now follows that all Berge graphs are perfect.

5 Algorithmic considerations

Berge graphs can be recognized in $O(n^9)$ time (Chudnovsky, Cornuéjols, Liu, Seymour and Vušković 2005). By the Strong Perfect Graph Theorem, it follows that perfect graphs can be recognized in $O(n^9)$ time.

[24]Note that this means that chordal graphs are precisely the graphs that contain no holes.

Further, Grötschel, Lovász, and Schrijver (1981) showed that the following optimization problems can be solved in polynomial time for perfect graphs: Maximum Clique, Maximum Stable Set, Graph Coloring (i.e. Vertex Coloring), and Minimum Clique Cover. In fact, weighted versions of these problems can also be solved in polynomial time.