NDMI012: Combinatorics and Graph Theory 2

Lecture #6

Chordal graphs

Irena Penev

April 7, 2021

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3. chordal graphs.

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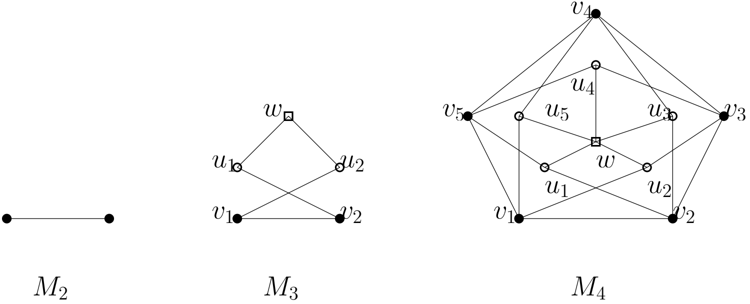
- Our goal is to construct a family of triangle-free graphs of arbitrarily large chromatic number.
- There are several known constructions; here, we give the one due to Mycielski (1955).

- Mycielski constructed a family of graphs $\{M_k\}_{k=2}^{\infty}$ as in the picture below (formal definition: Lecture Notes).



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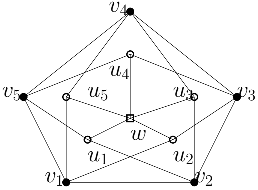

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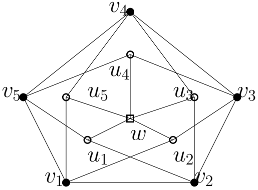

$\omega(M_{k+1}) = 2$: easy. (Details: Lecture Notes.)

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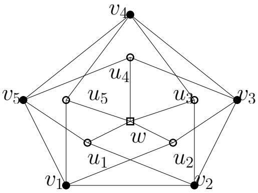

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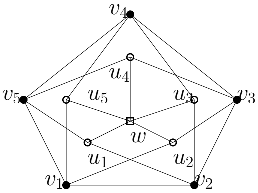

To see that $\chi(M_{k+1}) \leq k + 1$, properly color M_k with colors $1, \ldots, k$ (possible by the induction hypothesis), then color each u_i with the same color as v_i, and finally color w with color $k + 1$.

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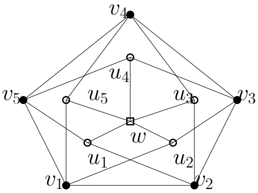

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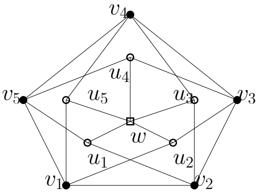

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Suppose now $\chi(M_{k+1}) \leq k$, and fix a proper coloring $c : V(M_{k+1}) \to \{1, \ldots, k\}$ of M_{k+1}. We will construct a proper coloring of M_k with only $k - 1$ colors, contrary to the induction hypothesis.

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For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

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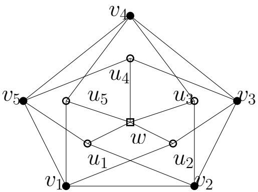

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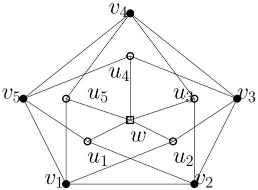

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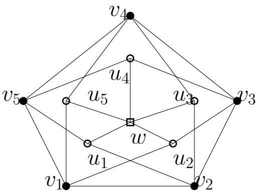

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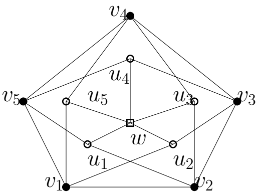

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  - The *girth* of a graph G that has at least one cycle is the length of the shortest cycle in G.
- Graphs of high girth are triangle-free, and so this result of Erdős is stronger than Theorem 1.2.

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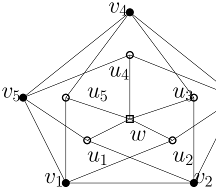
- This, however, turns out not to be a very interesting question.
- Indeed, suppose H is any graph at all, and let G be the disjoint union of H and $K_{\chi(H)}$.



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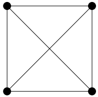

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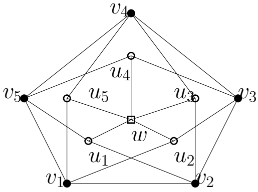

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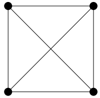

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- Then $\chi(G) = \omega(G)$, but we can say very little about the structure of G (since G was built starting from an arbitrary graph H).

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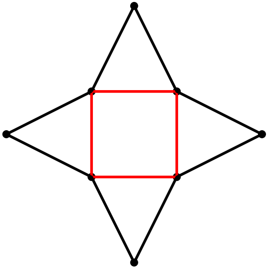
- Since every graph is an induced subgraph of itself, we see that every perfect graph G satisfies $\chi(G) = \omega(G)$.
- Importantly, though, in a perfect graph, $\chi = \omega$ should hold not only for the graph itself, but also for all its induced subgraphs.

Part III: Chordal graphs

Definition

A graph is *chordal* (or *triangulated*) if every cycle of length strictly greater than three has a chord (a *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle).

- In other words, a graph is *chordal* if it contains no induced cycles of length at least four.



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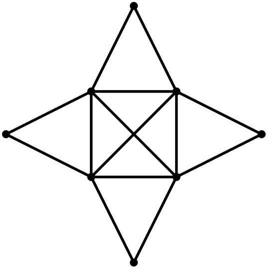

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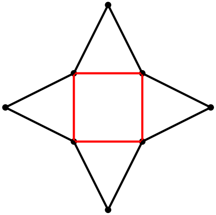

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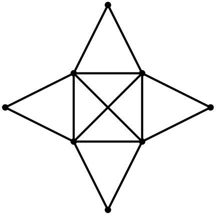

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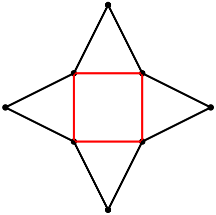
- Note that all induced subgraphs of a chordal graph are chordal.



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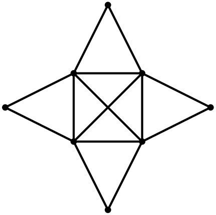

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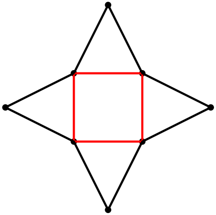

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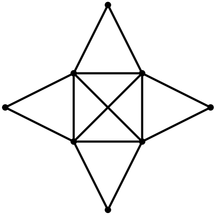

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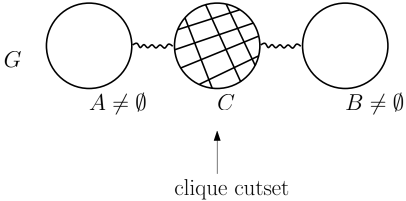
- As we shall see, there are efficient algorithms for recognizing chordal graphs and for solving the vertex coloring and related optimization problems on chordal graphs.

Definition

A *cutset* of a graph is a set of vertices whose deletion yields a disconnected graph. More precisely, a *cutset* of a graph G is a (possibly empty) set $S \subsetneq V(G)$ such that $G \setminus S$ is disconnected. A *clique-cutset* is a cutset that is a clique, that is, a *clique-cutset* of a graph G is a clique $C \subsetneq V(G)$ of G such that $G \setminus C$ is disconnected.



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Let G be a chordal graph that is not complete, let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x and y. Then S is a clique of G.

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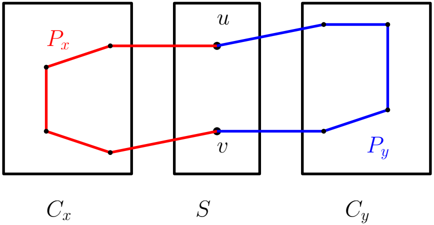
Proof (outline). Suppose that S is not a clique, and let u and v be two nonadjacent vertices of S. Let C_x be the component of $G \setminus S$ that contains x, and let C_y be the component of $G \setminus S$ that contains y.



Now $P_x \cup P_y$ is an induced cycle of length at least four in G, a contradiction.

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Proof. Let G be a chordal graph that is not complete. Let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x from y. By Lemma 3.1, S is a clique. It follows that S is a clique-cutset of G.

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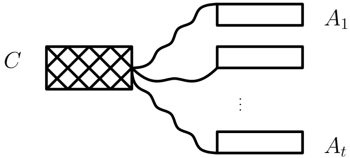

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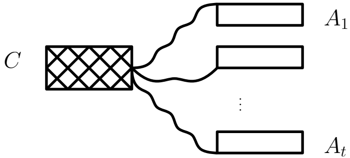

Note that every clique of G is in fact a clique of one of $G_1, \ldots, G_t$, and it follows that $\omega(G) = \max\{\omega(G_1), \ldots, \omega(G_t)\}$.

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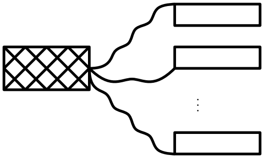

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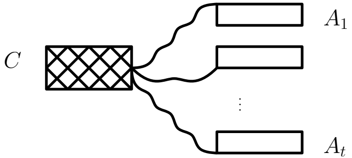

Note that every clique of G is in fact a clique of one of $G_1, \dots, G_t$, and it follows that $\omega(G) = \max\{\omega(G_1), \dots, \omega(G_t)\}$. On the other hand, by Lemma 2.1 from Lecture Notes 4, we have that $\chi(G) = \max\{\chi(G_1), \dots, \chi(G_t)\}$. Finally, for all $i \in \{1, \dots, t\}$, the induction hypothesis guarantees that $\chi(G_i) = \omega(G_i)$.

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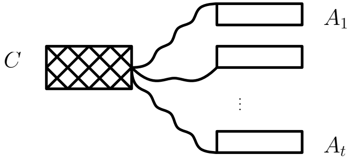
Proof (continued).



So,

$$\begin{aligned} \chi(G) &= \max\{\chi(G_1), \ldots, \chi(G_t)\} \\ &= \max\{\omega(G_1), \ldots, \omega(G_t)\} \\ &= \omega(G), \end{aligned}$$

which is what we needed.

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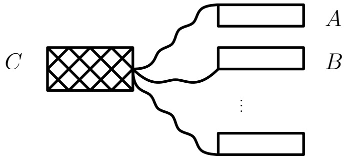
Proof (outline). By induction on the number of vertices. Obviously true for complete graphs. If G is chordal, but not complete, then by induction hypothesis, $G[A \cup C]$ has a simplicial vertex $a \in A$, and $G[B \cup C]$ has a simplicial vertex $b \in B$. Now a and b are non-adjacent simplicial vertices of G.



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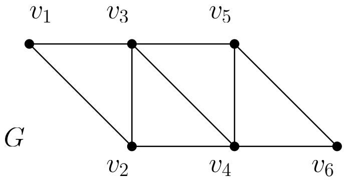
- For instance, $v_1, \dots, v_6$ is a simplicial elimination ordering of the graph G in the picture below.



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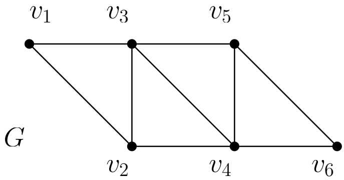
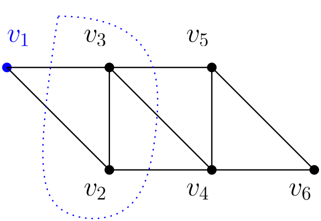
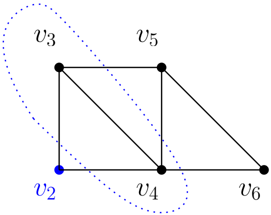


- Indeed (next slide).



$G[v_1, v_2, v_3, v_4, v_5, v_6]$



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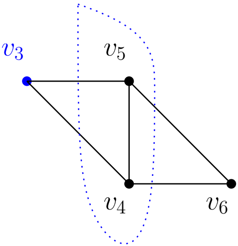

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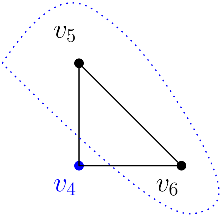

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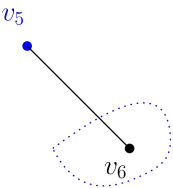

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Theorem 3.5 [Fulkerson and Gross, 1965]

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(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline).

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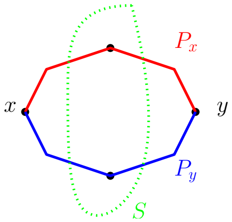
(iii) $\Rightarrow$ (i): If (i) is false, then (iii) is false. (Details: Lecture Notes.)



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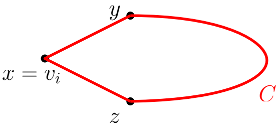

Then $yz \in E(G)$, and so C is a triangle. Thus, G is chordal.

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- In fact, chordal graphs can be recognized in $O(n + m)$ time using the so called Lexicographic breadth-first-search (LexBFS) due to Rose, Tarjan, and Lueker (1976), but we omit the details.

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- For the rest of this lecture, G is a chordal graph on n vertices, and $v_1, \dots, v_n$ is a simplicial elimination ordering on G.
- For each $i \in \{1, \dots, n\}$, set $X_i := N_G[v_i] \cap \{v_i, \dots, v_n\}$.
    - So, X_i is the closed neighborhood of v_i in the graph $G[v_i, \dots, v_n]$.

Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ such that $C = X_i$.[a]

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Theorem 3.7 [Fulkerson and Gross, 1965]

G has at most n maximal cliques. Furthermore, equality holds if and only if G is edgeless.

Proof. This follows from Lemma 3.6 (Details: Lecture Notes.)

Definition

A *clique cover* of a graph H is a partition of $V(H)$ into cliques. The *clique cover number* of H, denoted by $\overline{\chi}(H)$, is the smallest size of a clique cover of H; a *minimum clique cover* of H is a clique cover of size precisely $\overline{\chi}(H)$.

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- Since proper colorings correspond to partitions of the vertex set into stable sets (color classes), it is clear that every graph H satisfies $\overline{\chi}(H) = \chi(\overline{H})$ and $\alpha(H) \leq \overline{\chi}(H)$.

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- If $V(G) = X_{i_1} \cup \dots \cup X_{i_{j-1}}$, then we set $t = j - 1$, and we terminate the sequence; otherwise, we let $i_j \in \{1, \dots, n\}$ be the smallest index such that $v_{i_j} \notin X_{i_1} \cup \dots \cup X_{i_{j-1}}$.

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- Set $Y_1 := X_{i_1}$, and for all $j \in \{2, \ldots, t\}$, set $Y_j := X_{i_j} \setminus (Y_1 \cup \cdots \cup Y_{j-1})$.

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Theorem 3.8 [Gavril, 1972]

The set $\{v_{i_1}, \dots, v_{i_t}\}$ is a maximum stable set of G, and $(Y_1, \dots, Y_t)$ is a minimum clique cover of G.

Proof (outline).

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Proof (outline). The fact that $\{v_{i_1}, \dots, v_{i_t}\}$ is a stable set and $(Y_1, \dots, Y_t)$ is a clique-cover of G follows from the construction. But now $t \leq \alpha(G) \leq \overline{\chi}(G) \leq t$, and so $\alpha(G) = \overline{\chi}(G) = t$. So, our stable set is maximum, and our clique cover is minimum.

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$.[a]

[a] So, we are using the reverse of our simplicial elimination ordering.

Proof.

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$.[a]

[a] So, we are using the reverse of our simplicial elimination ordering.

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Proof. Clearly, the greedy coloring produces a proper coloring of G. If we apply the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$, then when we reach a vertex v_i, the neighbors of v_i that have already been colored are precisely those from the clique $X_i \setminus \{v_i\}$,

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Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ such that $C = X_i$.

Theorem 3.8 [Gavril, 1972]

The set $\{v_{i_1}, \ldots, v_{i_t}\}$ is a maximum stable set of G, and $(Y_1, \ldots, Y_t)$ is a minimum clique cover of G.

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- Clearly, Lemma 3.6, Theorem 3.8, and Lemma 3.9 yield polynomial time algorithms for finding a maximum clique, a maximum stable set, a minimum clique-cover, and an optimal coloring of a chordal graph.