NDMI012: Combinatorics and Graph Theory 2

Lecture #5
Vertex and edge coloring: Brooks' theorem and Vizing's theorem

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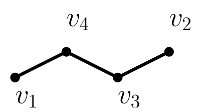
1 Vertex coloring: Brooks' theorem

A *greedy* coloring of a graph G with vertex ordering $V(G) = \{v_1, \dots, v_n\}$ is a coloring of G obtained as follows: for each $i \in \{1, \dots, n\}$, we assign to v_i the smallest positive integer that was not used on any smaller-indexed neighbor of v_i.

For example, the greedy coloring applied to the graph below, with the ordering v_1, v_2, v_3, v_4, yields the coloring $c(v_1) = 1$, $c(v_2) = 1$, $c(v_3) = 2$, and $c(v_4) = 3$.



Note that the greedy coloring of a graph G always produces a proper coloring of G, but the coloring need not be optimal, i.e. it may use more than $\chi(G)$ colors (indeed, this was the case in the example above).

Lemma 1.1. *Every graph G satisfies $\chi(G) \leq \Delta(G) + 1$.*

Proof. A greedy coloring of a graph G (using any ordering of $V(G)$) produces a proper coloring of G that uses at most $\Delta(G) + 1$ colors; so, $\chi(G) \leq \Delta(G) + 1$. □

If G is a complete graph or an odd cycle, then it is easy to see that $\chi(G) = \Delta(G)+1$, i.e. the inequality from Lemma 1.1 is an equality. However, as we shall see, if G is a connected graph other than a complete graph or odd cycle, then the inequality from Lemma 1.1 is strict, i.e. $\chi(G) \leq \Delta(G)$ (see Brooks' theorem below). First, we prove a technical lemma.

Lemma 1.2. *If G is connected and not regular, then $\chi(G) \leq \Delta(G)$.*

Proof. Let G be a connected graph that is not regular, and fix a vertex $v \in V(G)$ such that $d_G(v) \leq \Delta(G) - 1$. We order $V(G)$ according to the distance from v, that is, we list v first, then we list all vertices at distance one from v (in any order), then we list all vertices at distance two from v (in any order), etc. Let $v_1, \dots, v_n$ be the resulting ordering of G. We now color G greedily using the ordering $v_n, \dots, v_1$.[1] By construction, every vertex in the ordering $v_n, \dots, v_1$, other than the vertex v_1, has at least one neighbor to the right of it, and therefore at most $\Delta(G) - 1$ neighbors to the left of it in the ordering $v_n, \dots, v_1$. But since $d_G(v) \leq \Delta(G) - 1$, we see that $v_1 = v$ also has at most $\Delta(G) - 1$ neighbors to the left of it in the ordering $v_n, \dots, v_1$. So, our coloring of G uses at most $\Delta(G)$ colors, and we deduce that $\chi(G) \leq \Delta(G)$. □

Brooks' theorem. *Let G be a connected graph that is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.*

Proof. To simplify notation, we set $\Delta := \Delta(G)$. We must show that $\chi(G) \leq \Delta$.

Since G is connected and not complete, we see that $\Delta \geq 2$. Next, suppose that $\Delta = 2$. Since G is connected, it follows that G is either a path on at least two edges or a cycle. But by hypothesis, G is not an odd cycle, and so G is either a path on at least two edges or an even cycle. It is now obvious that $\chi(G) \leq \Delta$.

From now on, we assume that $\Delta \geq 3$. Note that this implies that $|V(G)| \geq 4$.[2] We may further assume that G is regular, for otherwise, we are done by Lemma 1.2.

Claim 1. If G has a clique-cutset, then $\chi(G) \leq \Delta$.

Proof of Claim 1. Suppose that G has a clique-cutset, and let C be a minimal clique-cutset of G. Let $A_1, \dots, A_t$ ($t \geq 2$) be the vertex sets of the components of $G \setminus C$. For all $i \in \{1, \dots, t\}$, let $G_i := G[A_i \cup C]$. By Lemma 2.1 from Lecture Notes 4, we have that

$$\chi(G) \quad = \quad \max\{\chi(G_1), \dots, \chi(G_t)\}.$$

Now, since G is connected, we know that C is non-empty. Further, by the minimality of C, we know that each vertex of C has a neighbor in each of

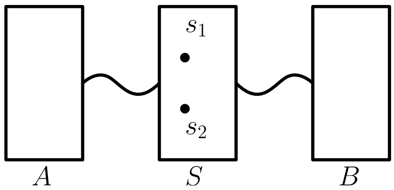
[1] Technically, we are applying the greedy coloring algorithm to the graph G with the ordering $u_1, \dots, u_n$, where $u_i = v_{n-i+1}$ for all $i \in \{1, \dots, n\}$. So, "smaller indexed" from the description of the greedy coloring algorithm refers to the indices of the u_i's, not the v_i's.

[2] Indeed, consider a vertex of degree Δ, plus all its neighbors.

$A_1, \dots, A_t$. This implies that $G_1, \dots, G_t$ are all connected and not regular.[3] But now Lemma 1.2 guarantees that $\chi(G_i) \le \Delta(G_i) \le \Delta$ for all $i \in \{1, \dots, t\}$. Consequently, $\chi(G) \le \Delta$. This proves Claim 1. ■

Claim 2. If G is not 3-connected, then $\chi(G) \le \Delta$.

Proof of Claim 2. Assume that G is not 3-connected; we must show that $\chi(G) \le \Delta$. We may assume that G does not have a clique-cutset, for otherwise, we are done by Claim 1. Since $|V(G)| \ge 4$, but G is not 3-connected, we see that there exists some $S \subseteq V(G)$ such that $|S| \le 2$ and $G \setminus S$ is disconnected. Since G does not admit a clique-cutset, we see that S is not a clique; consequently, $|S| = 2$ (say, $S = \{s_1, s_2\}$), and $s_1 s_2 \notin E(G)$. Let (A, B) be a partition of $V(G) \setminus S$ into non-empty sets such that there are no edges between A and B.



Note that each vertex of S has a neighbor both in A and in B (otherwise, s_1 or s_2 would be a cut-vertex of G, contrary to the fact that G has no clique-cutset). Furthermore, since $s_1 s_2 \notin E(G)$, and since G is Δ-regular, with $\Delta \ge 3$, we see that each of s_1, s_2 has at least two neighbors in at least one of A, B. So, by symmetry, there are two cases to consider:

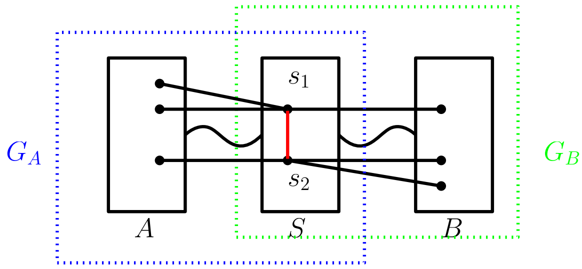
(i) s_1 has at least two neighbors in A, and s_2 has at least two neighbors in B;

(ii) s_1, s_2 each have exactly one neighbor in A.

Suppose first that (i) holds. Let $G_A := G[A \cup S] + s_1 s_2$ and $G_B := G[B \cup S] + s_1 s_2$. Then both G_A and G_B are connected, with $\Delta(G_A), \Delta(G_B) = \Delta$.[4] Furthermore, note that $d_{G_A}(s_2) \le d_G(s_2) - 1 = \Delta - 1$, and so G_A is not regular; thus, Lemma 1.2 guarantees that $\chi(G_A) \le \Delta(G_A) = \Delta$, and similarly, $\chi(G_B) \le \Delta$.

[3] Indeed, for all $i \in \{1, \dots, t\}$ and $a_i \in A_i$, we have that $d_{G_i}(a) = d_G(a) = \Delta$, whereas each $c \in C$ has a neighbor in $V(G) \setminus V(G_i)$ and consequently satisfies $d_{G_i}(c) \le d_G(c) - 1 \le \Delta - 1$. So, G_i is not regular.

[4] Indeed, for any $a \in A$, we have that $d_{G_A}(a) = d_G(a) = \Delta$, and $d_{G_A}(s_1), d_{G_A}(s_2) \le \Delta$; so, $\Delta(G_A) = \Delta$, and similarly, $\Delta(G_B) = \Delta$.



Now, note that S is a clique-cutset of $G + s_1s_2$. Lemma 2.1 from Lecture Notes 4 now implies that $\chi(G + s_1s_2) = \max\{\chi(G_A), \chi(G_B)\} \leq \Delta$,[5] and it follows that $\chi(G) \leq \Delta$.

Suppose now that (ii) holds. Note that this implies that each of s_1, s_2 has at least two neighbors in B. Let s_1' be the unique neighbor of s_1 in A. Set $S' := \{s_1', s_2\}$, $A' := A \setminus \{s_1'\}$, and $B' := B \cup \{s_1\}$. Since G is Δ-regular, with $\Delta \geq 3$, we know that s_1' has at least three neighbors; since all neighbors of s_1' are in $A \cup S$, and $|S| = 2$, we see that s_1' has a neighbor in A. It follows that $A' \neq \emptyset$. Now S' separates $A' \neq \emptyset$ from $B' \neq \emptyset$. Further, if $s_1's_2 \in E(G)$, then S' is a clique-cutset of G, a contradiction. So, we may assume that $s_1's_2 \notin E(G)$. Since s_1' has at least three neighbors, and they all belong to $A \cup S$, we deduce that s_1' in fact has at least two neighbors in A'. But now if we consider S', A', B' instead of S, A, B, we are back in case (i), and so an argument analogous to the above guarantees that $\chi(G) \leq \Delta$. This proves Claim 2. ■

In view of Claim 2, we may now assume that G is 3-connected. Since G is connected and not complete, G has two vertices, call them u and v, at distance two from each other; let w be a common neighbor of u and v. Since G is 3-connected, we know that $G' := G \setminus \{u, v\}$ is connected. We now order $V(G')$ according to the distance from w, that is, we list w first, then we list all vertices at distance one from w in G' (in any order), then we list all vertices at distance two from w in G' (in any order), etc. Finally, we list u, v at the end of our list. This produces an ordering $v_1, \dots, v_n$ of $V(G)$ (with $v_1 = w$, $v_{n-1} = u$, and $v_n = v$). We now color G greedily using the ordering $v_n, \dots, v_1$. All vertices in the ordering $v_n, \dots, v_1$ other than the vertex v_1 have at least one neighbor to the right, and therefore at most $\Delta - 1$ neighbors to the left; so, all vertices other than v_1 get a color from the set $\{1, \dots, \Delta\}$. But v_1 has exactly Δ neighbors, and two of those (namely, $v_{n-1} = u$ and $v_n = v$) got assigned the same color (namely, color 1) by our greedy coloring. So, v_1 also got assigned a color from the color set $\{1, \dots, \Delta\}$. It follows that $\chi(G) \leq \Delta$. □

[5] We are using the fact that A is the union of the vertex sets of some components of $G \setminus S = (G + s_1s_2) \setminus S$, whereas B is the union of the vertex sets of the remaining components of $G \setminus S = (G + s_1s_2) \setminus S$.

2 Eulerian graphs

An *Euler circuit* (or *Eulerian circuit*) is a walk in the graph that passes through every edge exactly once and comes back to the origin vertex. A graph is *Eulerian* if it has an Eulerian circuit. The following theorem was proven in Discrete Mathematics.

Theorem 2.1. *A connected graph is Eulerian if and only if it has no vertices of odd degree.*

3 Vizing's theorem

A *k-edge-coloring* of a graph G is a mapping $c : E(G) \to C$, with $|C| = k$. Elements of C are called *colors*. An edge-coloring is *proper* if for any two distinct edges e and f that share an endpoint, we have that $c(e) \neq c(f)$.

A graph G is *k-edge-colorable* if it has a proper k-edge-coloring.

The *edge chromatic number* (or *chromatic index*) of a graph G, denoted by $\chi'(G)$, is the minimum k such that G is k-edge-colorable.

Clearly, in any proper edge-coloring of a graph G, all edges incident with the same vertex must receive a different color; consequently, $\chi'(G) \geq \Delta(G)$.

Note that any k-edge-coloring (not necessarily proper) can be represented by a partition $\mathcal{C} = (E_1, \dots, E_k)$ of $E(G)$, where E_i denotes the subset of $E(G)$ assigned color i. (Sets $E_1, \dots, E_k$ are called *color classes*.) A proper k-edge-coloring is one where each E_i is a matching.

Lemma 3.1. *Every graph G satisfied $\chi'(G)\nu(G) \geq |E(G)|$. Consequently, if G has at least one edge, then $\chi'(G) \geq \left\lceil \frac{|E(G)|}{\nu(G)} \right\rceil$.*[6]

Proof. Let G be a graph, and let $k = \chi'(G)$. Let $(E_1, \dots, E_k)$ be a proper edge-coloring of G. Then

$$\begin{aligned}
|E(G)| &= \sum_{i=1}^{k} |E_i| && \text{because } (E_1, \dots, E_k) \text{ is a partition of } E(G) \\
&\leq \sum_{i=1}^{k} \nu(G) && \text{because } E_1, \dots, E_k \text{ are matchings of } G \\
&= k\nu(G) \\
&= \chi'(G)\nu(G).
\end{aligned}$$

[6]Recall that $\nu(G)$ is the matching number of G, i.e. the maximum size of a matching in G.

This proves that $\chi'(G)\nu(G) \geq |E(G)|$. If G has at least one edge, then clearly, $\nu(G) \geq 1$, and we deduce that $\chi'(G) \geq \frac{|E(G)|}{\nu(G)}$; since $\chi'(G)$ is an integer, it follows that $\chi'(G) \geq \left\lceil \frac{|E(G)|}{\nu(G)} \right\rceil$. □

Given a (not necessarily proper) edge-coloring of a graph G, we say that color i is *represented* at a vertex v of G if some edge incident with v has color i.

Lemma 3.2. *Let G be a connected graph that is not an odd cycle. Then G has a (not necessarily proper) 2-edge-coloring in which both colors are represented at each vertex of degree at least 2.*

Proof. We may assume that $\Delta(G) \geq 2$, for otherwise there is nothing to show. By hypothesis, G is connected and not an odd cycle; consequently, if G is 2-regular, then G is an even cycle.

Suppose first that G is Eulerian. Then (by Theorem 2.1) all vertices of G are of even degree. If G has a vertex of degree at least four, then let v_0 be such a vertex, and otherwise let v_0 be any vertex. (Note that in the latter case, G is 2-regular, and therefore, by the above, G is an even cycle.) Let $v_0, e_1, v_1, e_2, v_2, \dots, v_0$ be an Euler circuit of G. Let E_1 be the set of odd indexed edges, and let E_2 the set of even indexed edges. If G is an even cycle, then clearly, the edge-coloring (E_1, E_2) satisfies the lemma. Otherwise, v_0 is of degree at least four, and the edge-coloring (E_1, E_2) has the desired property since each vertex of G is an internal vertex of $v_0, e_1, v_1, e_2, v_2, \dots, v_0$.

So we may assume that G is not Eulerian. Construct G^* by adding a new vertex v^* and joining it to each vertex of odd degree in G. Then by Theorem 2.1, G^* is Eulerian.[7] Now, let $v_0, e_1, v_1, e_2, v_2, \dots, v_0$, with $v_0 = v^*$, be an Euler circuit of G^*. Let E_1 be the set of odd indexed edges, and let E_2 the set of even indexed edges. Then the edge-coloring $\Big(E_1 \cap E(G), E_2 \cap E(G)\Big)$ of G has the desired property. □

Given a (not necessarily proper) k-edge-coloring $\mathcal{C}$ and a vertex v of G, we denote by $c_{\mathcal{C}}(v)$ the number of distinct colors represented at v. Note that $c_{\mathcal{C}}(v) \leq d_G(v)$ for all $v \in V(G)$. Furthermore, $\mathcal{C}$ is a proper k-edge-coloring if and only if $c_{\mathcal{C}}(v) = d_G(v)$ for every vertex $v \in V(G)$. A k-edge-coloring $\mathcal{C}'$ of G is an *improvement* of $\mathcal{C}$ if

$$\sum_{v \in V(G)} c_{\mathcal{C}'}(v) \quad > \quad \sum_{v \in V(G)} c_{\mathcal{C}}(v).$$

[7] Since G is connected and not Eulerian, we know that G has at least one vertex of odd degree. On the other hand, since $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$, we know that $\sum_{v \in V(G)} d_G(v)$ is even, and consequently, G has an even number of vertices of odd degree. So, v^* has even degree, strictly greater than two. We now see that G^* is connected, and that all vertices of G^* have even degree. So, by Theorem 2.1, G^* is Eulerian.

An *unimprovable* k-edge-coloring is one that cannot be improved.

Note that any proper edge-coloring of a graph G is unimprovable. However, the converse does not hold in general.

Lemma 3.3. *Let $\mathcal{C} = (E_1, \dots, E_k)$ be an unimprovable k-edge-coloring of a graph G. If there is a vertex u of G and colors i and j such that i is not represented at u and j is represented at least twice at u, then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.*[8]

Proof. Let H be the component of $G[E_i \cup E_j]$ that contains u. Suppose that H is not an odd cycle. Then by Lemma 3.2, H has a 2-edge-coloring in which both colors are represented at every vertex of degree at least 2 in H. Recolor the edges of H with colors i and j in this way to get a new k-edge-coloring $\mathcal{C}' = (E'_1, \dots, E'_k)$ of G. To simplify notation, set $c = c_{\mathcal{C}}$ and $c' = c_{\mathcal{C}'}$. By construction, we have that $c(v) \leq c'(v) \leq c(v) + 1$ for all $v \in V(G)$, and that $c'(u) = c(u) + 1$. It follows that $\sum\limits_{v \in V(G)} c'(v) > \sum\limits_{v \in V(G)} c(v)$, that is, $\mathcal{C}'$ is an improvement of $\mathcal{C}$. But this contradicts the assumption that $\mathcal{C}$ is unimprovable. □

Theorem 3.4. *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Proof. Let G be a bipartite graph, and let $\Delta := \Delta(G)$. Clearly, $\chi'(G) \geq \Delta$, and we need only show that $\chi'(G) \leq \Delta$. Let $\mathcal{C} = (E_1, \dots, E_\Delta)$ be an unimprovable Δ-edge-coloring of G. Suppose that $\mathcal{C}$ is not a proper edge-coloring of G. Then there exists a vertex $u \in V(G)$ such that some color j is represented at least twice at u, and (consequently) some color i is not represented at u. But now by Lemma 3.3, the component of $G[E_i \cup E_j]$ that contains u is an odd cycle, contrary to the fact that bipartite graphs contain no odd cycles. So, $\mathcal{C}$ is a proper Δ-edge-coloring of G, and it follows that $\chi'(G) \leq \Delta$. □

Vizing's theorem. *Every graph G satisfies $\chi'(G) \leq \Delta(G) + 1$.*[9]

Proof. Let $\Delta = \Delta(G)$. Suppose that $\chi'(G) > \Delta + 1$. Let $\mathcal{C} = (E_1, \dots, E_{\Delta+1})$ be an unimprovable $(\Delta + 1)$-edge-coloring, and set $c = c_{\mathcal{C}}$. Since no vertex of G has degree greater than Δ, and since we have $\Delta + 1$ colors, we know that for each vertex of G, at least one of our $\Delta + 1$ colors is not represented at that vertex. On the other hand, since $\chi'(G) > \Delta + 1$, we know that $\mathcal{C}$ is not a proper edge-coloring of G, and consequently, at some vertex of G, some color is represented at least twice.

Let vertex $u \in V(G)$ and colors $i_0, i_1 \in \{1, \dots, \Delta + 1\}$ be such that i_0 is not represented at u, and i_1 is represented at least twice at u. Let uv_1 have color i_1, and let i_2 be a color that is not represented at v_1. (Clearly,

[8] Here, $G[E_i \cup E_j]$ is the graph with vertex set $V(G)$ and edge set $E_i \cup E_j$.

[9] As usual, we consider only simple graphs. Vizing's theorem fails if G is not simple!

$i_1 \neq i_2$.) Color i_2 must be represented at u, since otherwise, recoloring uv_1 with i_2 would yield an improvement of $\mathcal{C}$. So some edge uv_2 has color i_2; let i_3 be a color that is not represented at v_2. (Clearly, $i_2 \neq i_3$.) Color i_3 must be represented at u, since otherwise recoloring uv_1 with i_2 and uv_2 with i_3 would yield an improvement of $\mathcal{C}$. So some edge uv_3 has color i_3. Now, we have only a finite number of colors at our disposal, and so continuing in this way, we eventually start to repeat colors. More formally, we can construct a sequence $v_1, v_2, \dots, v_\ell$ of vertices and a sequence $i_1, i_2, \dots, i_\ell, i_{\ell+1}$ of colors such that all the following are satisfied:

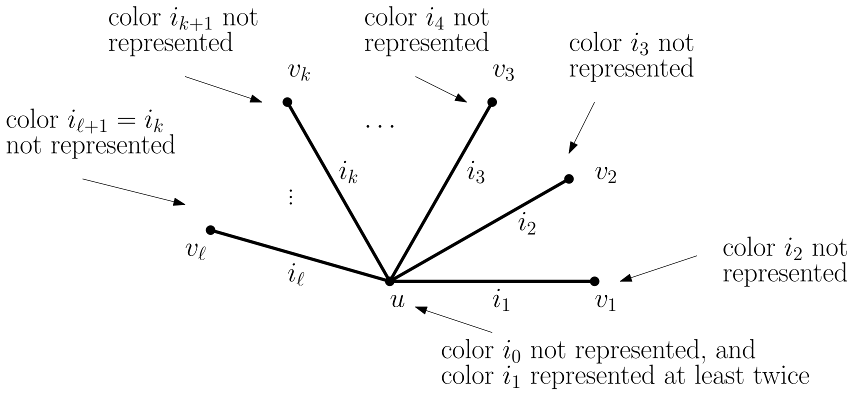
(a) color i_1 is represented at least twice at u;

(b) for all $j \in \{1, \dots, \ell\}$, edge uv_j has color i_j;

(c) for all $j \in \{1, \dots, \ell\}$, color i_{j+1} is not represented at v_j;

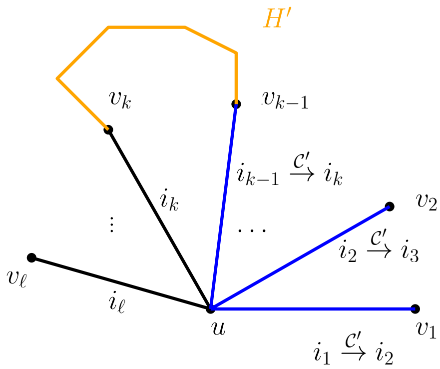
(d) colors $i_1, \dots, i_\ell$ are pairwise distinct;

(e) there exists some $k \in \{1, \dots, \ell\}$ such that $i_k = i_{\ell+1}$.

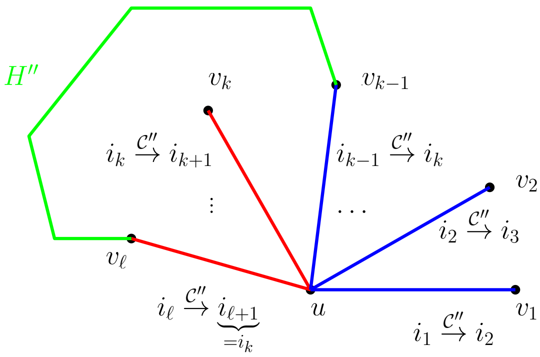


Note that (b) and (c) imply that $i_j \neq i_{j+1}$ for all $j \in \{1, \dots, \ell\}$; in particular then, $k \leq \ell - 1$. Further, (b) and (d) imply that vertices $v_1, \dots, v_\ell$ are pairwise distinct.

Let $\mathcal{C}' = (E'_1, \dots, E'_{\Delta+1})$ be the following recoloring of G: for $j = 1, \dots, k-1$, recolor uv_j with i_{j+1}. Set $c' = c_{\mathcal{C}'}$. Then $c'(v) \geq c(v)$ for every $v \in V(G)$; thus, since $\mathcal{C}$ is an unimprovable $(\Delta+1)$-edge-coloring of G, so is $\mathcal{C}'$. Further, by construction, under the coloring $\mathcal{C}'$, color i_0 is not represented at u, and color i_k is represented at least twice at u. (Note that if $k = 1$, then $\mathcal{C}' = \mathcal{C}$ and $i_k = i_1$. In this case, $i_k = i_1$ is still represented twice at u, by the choice of i_1.) Let H' be the component of $G[E'_{i_0} \cup E'_{i_k}]$ that contains u. By Lemma 3.3, H' is an odd cycle.



Let $\mathcal{C}'' = (E''_1, \dots, E''_{\Delta+1})$ be the following recoloring of G: for $j = 1, \dots, \ell$, recolor uv_j with i_{j+1}; since $i_{\ell+1} = i_k$, we see that uv_ℓ was recolored with i_k. Set $c'' = c_{\mathcal{C}''}$. Then $c''(v) \geq c(v)$ for every $v \in V(G)$; thus, since $\mathcal{C}$ is an unimprovable $(\Delta+1)$-edge-coloring of G, so is $\mathcal{C}''$. Further, under the coloring $\mathcal{C}'$, color i_0 is not represented at u, and color i_k is represented at least twice at u. Let H'' be the component of $G[E''_{i_0} \cup E''_{i_k}]$ that contains u. By Lemma 3.3, H'' is an odd cycle.

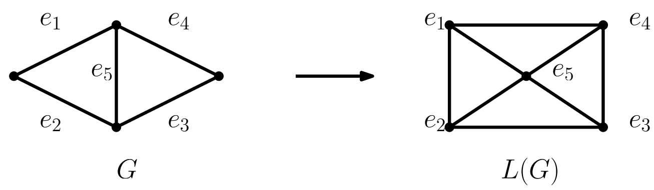


Note that the colorings $\mathcal{C}'$ and $\mathcal{C}''$ disagree only on edges $uv_k, \dots, uv_{\ell-1}, uv_\ell$. Further, exactly one edge (namely, uv_k) from $uv_k, \dots, uv_{\ell-1}, uv_\ell$ belongs to the cycle H', and consequently, the path $H' - uv_k$ is a subgraph of H''. Now, since H'' is a cycle, and v_k is an endpoint of the path $H' - uv_k$, we see that v_k must have a neighbor v'_k in the H'' that is different from its (unique) neighbor in $H' - uv_k$. But uv_k is colored $k+1$ by $\mathcal{C}''$, and so $v'_k \neq u$, and it follows that the edge $v_k v'_k$ does not belong to $uv_k, \dots, uv_{\ell-1}, uv_\ell$. But then $v_k v'_k$ is colored identically by $\mathcal{C}'$ and $\mathcal{C}''$, and we deduce that $v_k v'_k$ is an edge of H', which is impossible since $v'_k \notin V(H')$. □

Corollary 3.5. *Every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

We note that it is NP-complete to decide whether $\chi' = \Delta$ (even when $\Delta = 3$). We omit the details.

Finally, we remark that there is a relationship between vertex coloring and edge-coloring, as follows. Given a graph G, the *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$, in which distinct $e, f \in E(G)$ are adjacent if and only if they share an endpoint in G. An example is shown below.



Obviously, $\chi(L(G)) = \chi'(G)$.

Recall that for a graph G, the *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G.

Lemma 3.6. *Every graph G satisfies $\chi(L(G)) \leq \omega(L(G)) + 1$.*

Proof. Let G be a graph. Then clearly, $\chi(L(G)) = \chi'(G)$. Furthermore, for any vertex v, the set of all edges incident with v in G is a clique of size $d_G(v)$ in $L(G)$; consequently, $\omega(L(G)) \geq \Delta(G)$. But now

$$\begin{aligned} \chi(L(G)) &= \chi'(G) \\ &\leq \Delta(G) + 1 \qquad \text{by Vizing's theorem} \\ &\leq \omega(L(G)) + 1. \end{aligned}$$

□