# NDMI012: Combinatorics and Graph Theory 2

## Lecture #4

## Minors and planar graphs (part II)

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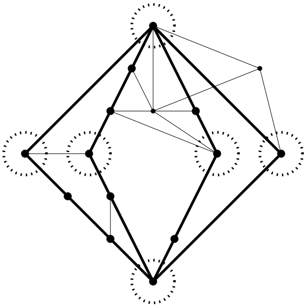
March 24, 2021

## Definition

A graph $H$ is a *topological minor* of a graph $G$, and we write $H \preceq_t G$, if $G$ contains some subdivision of $H$ as a subgraph. The vertices of this subdivision that correspond to the vertices of $H$ are called *branch vertices*.

- The graph below contains $K_{2,4}$ as a topological minor.

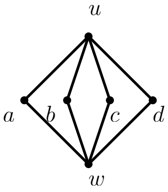


**Definition**

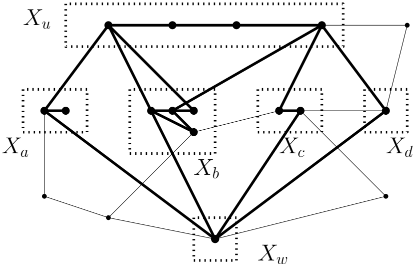
A graph $H$ is a *minor* of a graph $G$, and we write $H \preceq_m G$, if there exists a family $\{X_v\}_{v \in V(H)}$ of pairwise disjoint, non-empty subsets of $V(G)$, called *branch sets*, such that

- $G[X_v]$ is connected for all $v \in V(H)$, and
- for all $uv \in E(H)$, there is an edge between $X_u$ and $X_v$ in $G$.

- For example, the graph below (on the right) contains $K_{2,4}$ as a minor.




$K_{2,4}$



## Kuratowski's theorem [Kuratowski, 1930; Wagner, 1937]

Let $G$ be a graph. Then the following are equivalent:

(a) $G$ is planar;

(b) $G$ contains neither $K_5$ nor $K_{3,3}$ as a minor;

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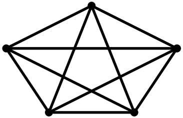
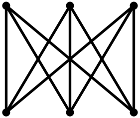


$K_5$



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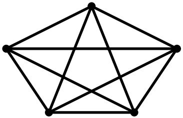

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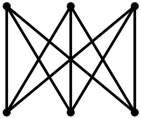

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- We proved "(a) $\implies$ (b)" and "(b) $\iff$ (c)" in the previous lecture.

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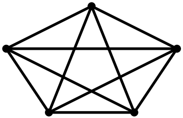

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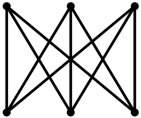

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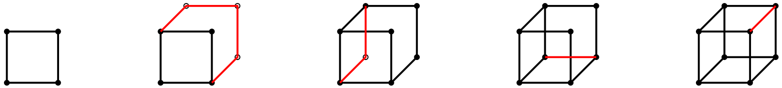
- We proved "(a) $\Longrightarrow$ (b)" and "(b) $\Longleftrightarrow$ (c)" in the previous lecture.
- In this lecture, we prove "(b) $\Longrightarrow$ (a)."

- A *path addition* (sometimes called *ear addition*) to a graph $H$ is the addition to $H$ of a path between two distinct vertices of $H$ in such a way that no internal vertex and no edge of the path belongs to $H$.



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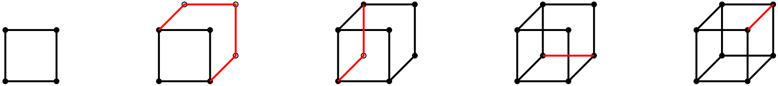

**The Ear Lemma**

A graph is 2-connected if and only if it is a cycle or can be obtained from a cycle by repeated ear addition.

*Proof.* Combinatorics & Graph Theory 1.

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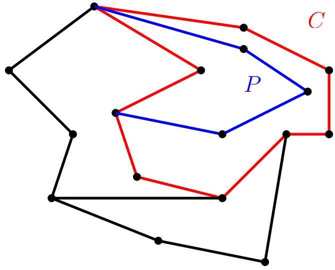
## Lemma 1.2

For any plane drawing of a planar 2-connected graph $G$, the boundary of each face is a cycle of $G$.

*Proof.* Lecture Notes (using the Ear Lemma).



**Lemma 1.3**

Let $G$ be a 3-connected graph that contains neither $K_5$ nor $K_{3,3}$ as a minor. Then $G$ is planar.

*Proof (outline).*

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*Proof (outline).* We may assume inductively that the lemma is true for graphs on fewer than $|V(G)|$ vertices, that is, that for all 3-connected graphs $H$ with $|V(H)| < |V(G)|$ and $K_5, K_{3,3} \not\preceq_m H$, we have that $H$ is planar.

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Since $G$ is 3-connected, we know that either $G \cong K_4$ or $|V(G)| > 4$.

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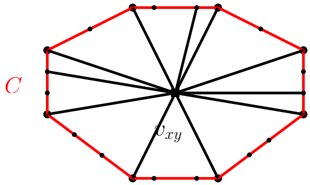
## Lemma 1.3

Let $G$ be a 3-connected graph that contains neither $K_5$ nor $K_{3,3}$ as a minor. Then $G$ is planar.

*Proof (outline, continued).* Fix a plane drawing of $H$. If we erase $v_{xy}$ and all the edges incident in it, we obtain a plane drawing of $H \setminus v_{xy}$. Now, let $f$ be the face of this drawing of $H \setminus v_{xy}$ such that $v_{xy}$ is in the interior of $f$. Since $H$ is 3-connected, $H \setminus v_{xy}$ is 2-connected; so, by Lemma 1.3, the boundary of $f$ is a cycle of $H \setminus v_{xy}$, say $C$. (Note that $C$ is also a cycle of $H$ and of $G$.)



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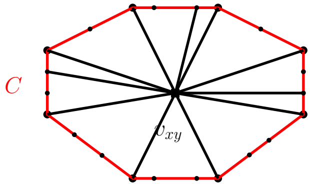

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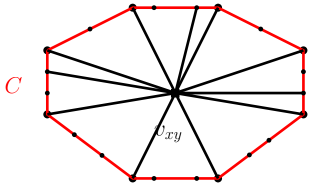


Then $N_H(v_{xy}) \subseteq V(C)$, and consequently, $N_G(x) \subseteq \{y\} \cup V(C)$ and $N_G(y) \subseteq \{x\} \cup V(C)$.

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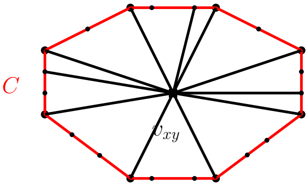


Then $N_H(v_{xy}) \subseteq V(C)$, and consequently, $N_G(x) \subseteq \{y\} \cup V(C)$ and $N_G(y) \subseteq \{x\} \cup V(C)$. Since $G$ is 3-connected, we know that $\delta(G) \geq 3$, and in particular, $d_G(x) \geq 3$; so, since $N_G(x) \subseteq \{y\} \cup V(C)$, $x$ has at least two neighbors in $V(C)$.

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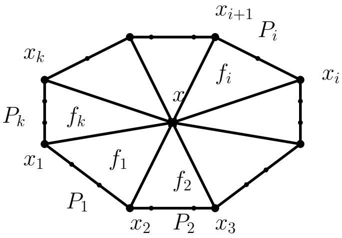
Let $G$ be a 3-connected graph that contains neither $K_5$ nor $K_{3,3}$ as a minor. Then $G$ is planar.

*Proof (outline, continued).* Let $x_1, \ldots, x_k$ be the neighbors of $x$ in $V(C)$, listed in cyclical order (along the cycle $C$). For each $i \in \{1, \ldots, k\}$, let $P_i$ be the path from $x_i$ to $x_{i+1}$ (we consider $x_{k+1} = x_1$) along $C$.



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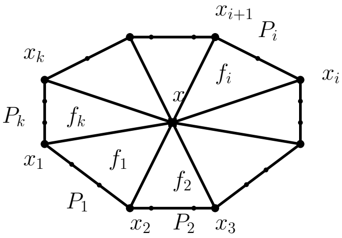


If for some $i \in \{1, \ldots, k\}$, we have that $N_G(y) \subseteq \{x\} \cup V(P_i)$, then $G$ is planar (details: Lecture Notes).

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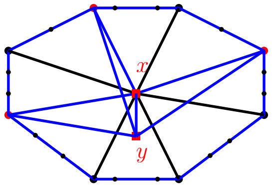
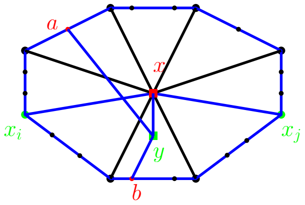
*Proof (outline, continued).* So, suppose that for all $i \in \{1, \ldots, k\}$, we have that $N_G(y) \not\subseteq \{x\} \cup V(P_i)$. Then either $x$ and $y$ have three common neighbors in $V(C)$, or $y$ has two neighbors $a, b \in V(C)$ that are separated in $C$ by two neighbors of $x$, say $x_i$ and $x_j$.



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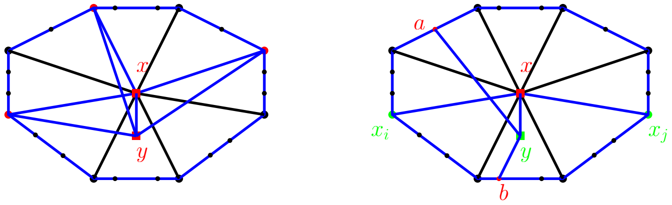


But then $G$ contains $K_5$ or $K_{3,3}$ as a topological minor, a contradiction.

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*Proof (outline).* We may assume inductively that for all graphs $H$ on fewer than $|V(G)|$ vertices, if $K_5, K_{3,3} \not\preceq_m H$, then $H$ is planar.

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Suppose first that $G$ is disconnected, and let $G_1, \dots, G_t$ be the components of $G$. Then by the induction hypothesis, $G_1, \dots, G_t$ are all planar. We obtain a plane drawing of $G$ by drawing $G_1, \dots, G_t$ in the plane side by side.

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Let $G$ be a graph that contains neither $K_5$ nor $K_{3,3}$ as a minor. Then $G$ is planar.

*Proof (outline, continued).* Next, suppose that $G$ is connected, but not 2-connected.

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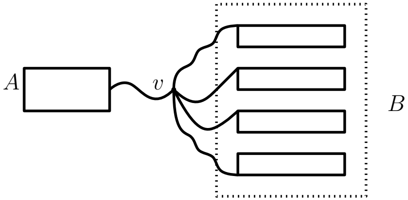

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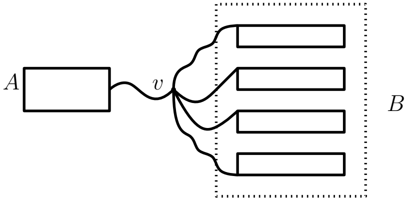


By the induction hypothesis, $G_A$ and $G_B$ are both planar.

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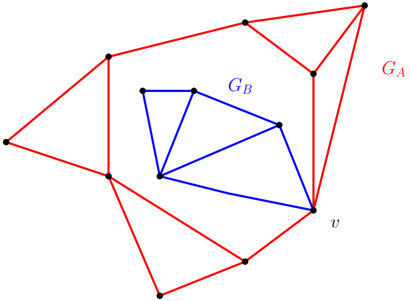
*Proof (outline, continued).* We draw $G_A$ in the plane without any edge crossings, and we let $f$ be some face of this drawing such that $v$ lies on the boundary of $f$. We then draw $G_B$ inside $f$, with $v$ coinciding in the drawing of $G_A$ and $G_B$.



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## Hajós' Conjecture

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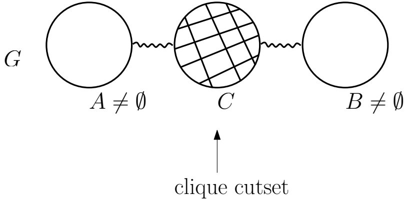
- Hajós' Conjecture is also true for $k = 4$, as we now show.
  - But first, we need a definition and a lemma.

## Definition

A *clique-cutset* of a graph $G$ is a clique $C \subsetneq V(G)$ of $G$ such that $G \setminus C$ is disconnected.[a]

[a] In particular, if $G$ is disconnected, then $\emptyset$ is a clique-cutset of $G$.



**Lemma 2.1**

Let $G$ be a graph, and let $C$ be a clique-cutset of $G$. Let $A_1, \ldots, A_t$ be the vertex sets of the components of $G \setminus C$. Then $\chi(G) = \max\{\chi(G[A_1 \cup C]), \ldots, \chi(G[A_t \cup C])\}$.

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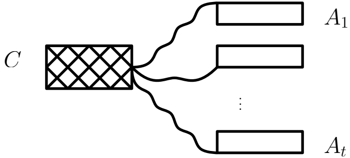


Obviously, $\max\{\chi_1, \dots, \chi_t\} \leq \chi(G)$.

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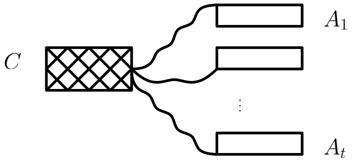


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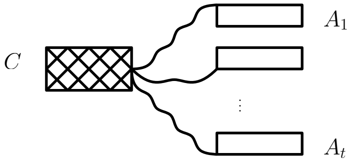


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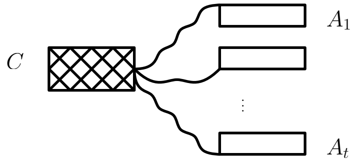


Obviously, $\max\{\chi_1, \dots, \chi_t\} \leq \chi(G)$. For the reverse inequality, fix optimal colorings of $G_1, \dots, G_t$ using the color set $\{1, \dots, \max\{\chi_1, \dots, \chi_t\}\}$. WMA these colorings agree on the clique $C$ (details: Lecture Notes). Now the union of these colorings is a proper coloring of $G$ that uses at most $\max\{\chi_1, \dots, \chi_t\}$ colors.

**Theorem 2.2 [Dirac, 1952]**

Every graph of chromatic number at least 4 contains $K_4$ as a topological minor.

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> ***Claim 1.*** *$G$ does not admit a clique-cutset. Furthermore, $G$ is 2-connected.*

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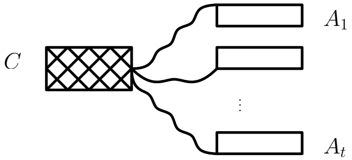
*Proof of Claim 1 (outline).* This follows from Lemma 2.1.



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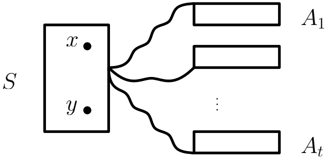

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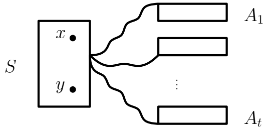

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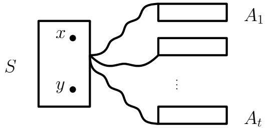


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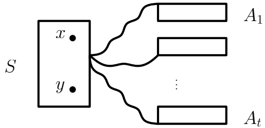


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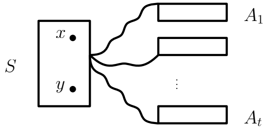


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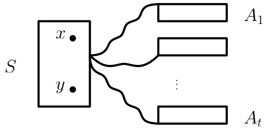

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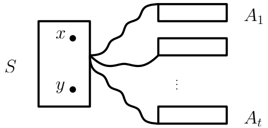


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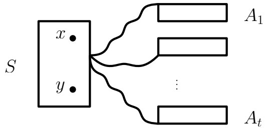


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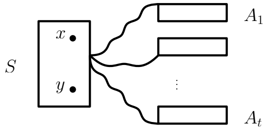


By symmetry, we may now assume that all 3-colorings of $G_1$ assign the same color to $x$ and $y$. But then $\chi(G_1 + xy) = 4$. So, by the induction hypothesis, we have that $K_4 \preceq_t G_1 + xy$.

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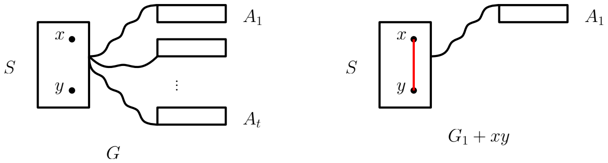


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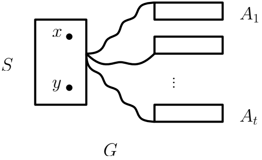


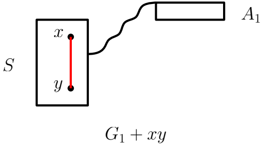


Now, since $G$ is 2-connected, we see that there exists an induced path $P$ in $G_2$ between $x$ and $y$. But now $G[A_1 \cup V(P)]$ is a subdivision of $G_1 + xy$, and so $G_1 + xy \preceq_t G$.

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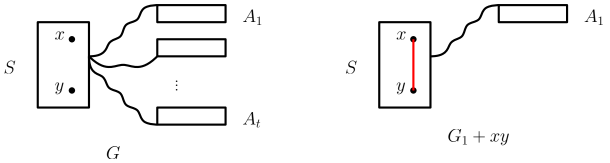


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*Proof (outline, continued).* In view of Claim 2, we may now assume that $G$ is 3-connected.

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*Proof of Claim 3.*

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So, we may assume that some two neighbors (call them $u_1$ and $u_2$) of $u$ are non-adjacent.

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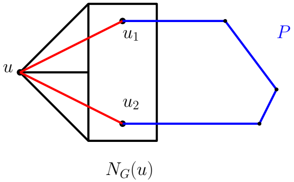
*Proof of Claim 3.* Since $G$ is 3-connected, we know that $G \setminus u$ is connected, and consequently, $G \setminus u$ contains a path $P$ between $u_1$ and $u_2$. But now $u - u_1 - P - u_2 - u$ is a cycle of length at least four in $G$.

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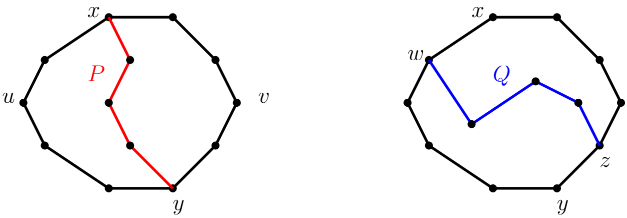

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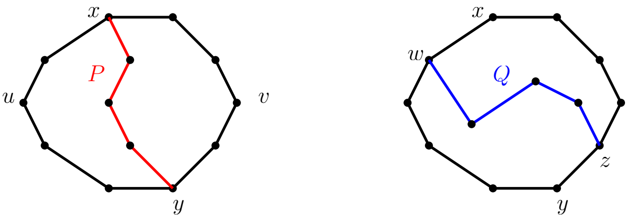
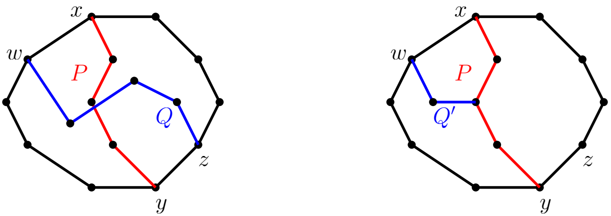



$P$ and $Q$ either do or do not intersect.

## Theorem 2.2 [Dirac, 1952]

Every graph of chromatic number at least 4 contains $K_4$ as a topological minor.

*Proof (outline, continued).* In either case, $G$ contains $K_4$ as a topological minor.



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- But for $k = 7$, it is false!

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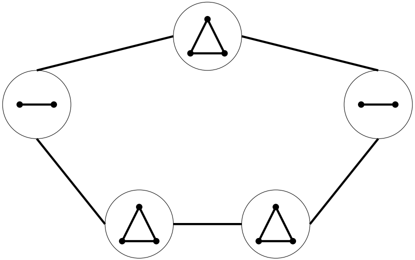

- The graph above has chromatic number 7, and yet it does not contain $K_7$ as a topological minor.

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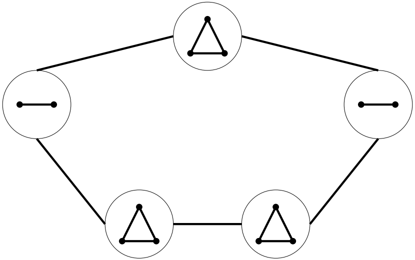

- The graph above has chromatic number 7, and yet it does not contain $K_7$ as a topological minor.
- For $k \geq 8$, we can obtain a counterexample to Hajós' Conjecture by adding $k - 7$ universal vertices to the graph above.

## Hajós' Conjecture

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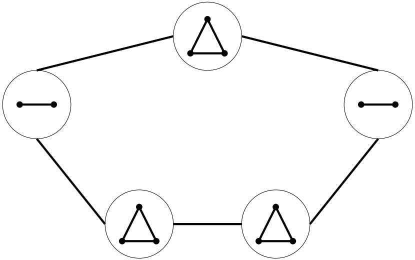

- The graph above has chromatic number 7, and yet it does not contain $K_7$ as a topological minor.
- For $k \geq 8$, we can obtain a counterexample to Hajós' Conjecture by adding $k - 7$ universal vertices to the graph above.
- Hajós' Conjecture is open for $k = 5$ and $k = 6$.

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### Hajós' Conjecture

For every positive integer $k$, every graph of chromatic number at least $k$ contains $K_k$ as a topological minor.

### Hadwiger's Conjecture

For every positive integer $k$, every graph of chromatic number at least $k$ contains $K_k$ as a minor.

- Since a topological minor is a special case of a minor, Hadwiger's Conjecture is weaker than Hajós' Conjecture. Thus, since Hajós' Conjecture is true for $k \leq 4$, Hadwiger's conjecture is also true for $k \leq 4$.

### Hadwiger's Conjecture

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**Hadwiger's Conjecture**

For every positive integer $k$, every graph of chromatic number at least $k$ contains $K_k$ as a minor.

- Hadwiger's Conjecture for $k = 5$ is equivalent to the famous Four Color Theorem.

**The Four Color Theorem [Appel and Haken, 1976]**

Every planar graph is 4-colorable.

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### The Four Color Theorem [Appel and Haken, 1976]

Every planar graph is 4-colorable.

- in 1993, Robertson, Seymour, and Thomas proved that Hadwiger's Conjecture is true for $k = 6$.
- For $k \geq 7$, the conjecture remains open.