NDMI012: Combinatorics and Graph Theory 2

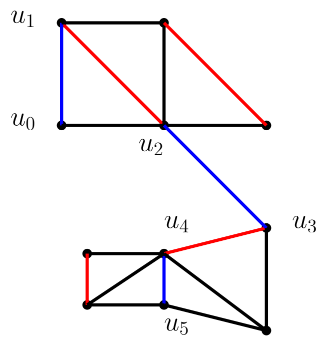
Lecture #2
Edmonds' Blossom algorithm

Irena Penev

Convention: In all our figures, edges of the matching in question are in red.[1]

1 M-augmenting paths

Let M be a matching in a graph G. An *M-alternating path* is a path $u_0, u_1, \dots, u_t$ in G such that every other edge of the path belongs to M (and the remaining edges do not). An *M-augmenting path* is an M-alternating path $u_0, u_1, \dots, u_t$ ($t \neq 0$) such that u_0, u_t are both unsaturated by M. For instance, in the picture below, $u_0, u_1, u_2, u_3, u_4, u_5$ is an M-augmenting path (as usual, the edges of the matching M are in red; the edges of the M-augmenting path that do not belong to M are in blue).



Lemma 1.1. *Let M be a matching in a graph G, and let $u_0, u_1, \dots, u_t$ be an M-augmenting path. Then t is odd and*

$$M' \quad := \quad \Big(M \setminus \{u_1u_2, u_3u_4, \dots, u_{t-2}u_{t-1}\}\Big) \cup \{u_0u_1, u_2u_3, \dots, u_{t-1}u_t\}$$

is a matching of G satisfying $|M'| = |M| + 1$.

[1] This is not a standard convention. We simply use it in these lecture notes.

Proof. This follows from the relevant definitions. □

Theorem 1.2. *[Berge, 1957] Let M be a matching in a graph G. Then M is a maximum matching of G if and only if G has no M-augmenting path.*

Proof. We will prove the contrapositive: the matching M is **not** maximum if and only if G has an M-augmenting path.

If G has an M-augmenting path, then Lemma 1.1 guarantees that M is not a maximum matching of G.

Suppose now that M is not a maximum matching, and let M' be matching of G such that $|M'| > |M|$. Let $F := M \Delta M'$,[2] and let H be the graph with vertex set $V(H) = V(G)$ and edge set $E(H) = F$. Clearly, $\Delta(H) \leq 2$.[3] So, H is the disjoint union of paths and cycles.

Now, since $|M'| > |M|$, some component P of H has more edges of M' than of M. If P is a cycle, then we see that some vertex of P is incident two edges of M', contrary to the fact that M' is a matching. So, P is a path, and it is easy to see that it is in fact an M-augmenting path in G.[4] □

2 Blossoms and stems

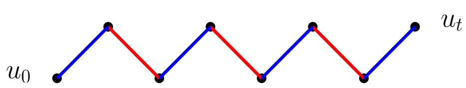
Our goal is to give a polynomial-time algorithm that finds a maximum matching in a graph. The basic idea is to start with an empty matching, and then repeatedly find augmenting paths and use them to find larger matchings (as in Lemma 1.1). We do this until no augmenting path remains, at which point Theorem 1.2 guarantees that our matching is maximum. We now need to show how we can find a maximum matching. In this section, we describe the basic tools that we need, and in the subsequent section, we describe the algorithm.

We begin with a definition. Suppose that M is a matching in a graph G. A *blossom* is a cycle $c_0, c_1, \dots, c_{2k}, c_0$ of length $2k+1$ (with $k \geq 1$) in

[2]By definition, $M \Delta M' = (M \setminus M') \cup (M' \setminus M)$.

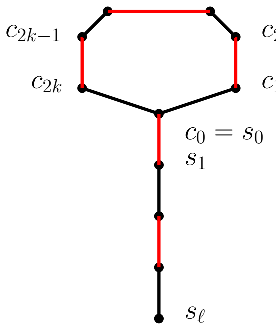
[3]Recall that $\Delta(H)$ is the maximum degree in H, i.e. $\Delta(H) = \max\{d_H(v) \mid v \in V(H)\}$. Let us check that $\Delta(H) \leq 2$. Since M and M' are matchings, we see that every vertex v of G is incident with at most one edge of M and at most one edge of M'. Since $V(H) = V(G)$ and $E(H) \subseteq M \cup M'$, it follows that every vertex of H is incident with at most two edges; thus, $\Delta(H) \leq 2$.

[4]Indeed, let P be of the form $u_0, u_1, \dots, u_t$. All edges of P are in $M \Delta M'$, and so since M and M' are both matchings, the edges of $M \setminus M'$ and $M' \setminus M$ alternate on P. Since P has more edges of M' than of M, we have that P has an odd number of edges (so, t is odd), and that $u_0u_1, u_2u_3, \dots, u_{t-1}u_t \in M' \setminus M$ and $u_1u_2, u_3u_4, \dots, u_{t-2}, u_{t-1} \in M \setminus M'$ (see the picture below; edges of $M \setminus M'$ are in red, and edges of $M' \setminus M$ are in blue).



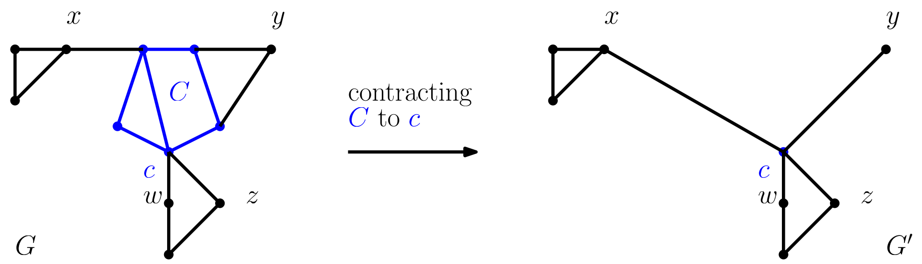
The fact that u_0, u_t are unsaturated by M follows from the construction of H, and from the fact that P is a component of H.

G in which edges $c_1c_2, c_3c_4, \ldots, c_{2k-1}c_{2k}$ belong to M, and the remaining $k+1$ edges do not belong to M. A *stem* for this blossom is an M-alternating path $s_0, \ldots, s_\ell$ such that $s_0 = c_0$ is the unique common vertex of the cycle $c_0, c_1, \ldots, c_{2k}, c_0$ and the path $s_0, \ldots, s_\ell$, and s_ℓ is unsaturated by M.[5] The union of a blossom and a corresponding stem is called a *flower*.[6] An example is shown below.



Next, let G be a graph, and let $C \subseteq V(G)$ and $c \in C$. We say that G' is the graph obtained form G by *contracting* C to c if

- $V(G') = V(G) \setminus (C \setminus \{c\}) = (V(G) \setminus C) \cup \{c\}$, and
- $E(G') = \left(\binom{V(G)\setminus C}{2} \cap E(G)\right) \cup \left\{xc \mid x \in V(G) \setminus C,\ \exists c' \in C \text{ s.t. } xc' \in E(G)\right\}$.



Lemma 2.1. *Let M be a matching in a graph G, and let $C = c_0, \ldots, c_{2k}, c_0$ be a blossom and $S = s_0, \ldots, s_\ell$ a corresponding stem (in particular, $c_0 = s_0$). Let G' be the graph obtained from G by contracting C to c_0,[7] and let $M' = M \setminus E(C)$. Then M' is a matching of G'. Furthermore, M is a maximum matching of G is and only if M' is a maximum matching of G'.*

---

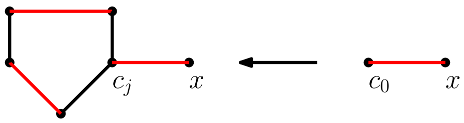
[5]Note that this implies that ℓ is even, furthermore, that either $\ell = 0$ and $c_0 = s_0$ is unsaturated by M, or $\ell \geq 2$ is even and $s_0s_1 \in M$.

[6]Note that there may be more than one stem for a fixed blossom. Nonetheless, all stems attach to the same vertex of the blossom.

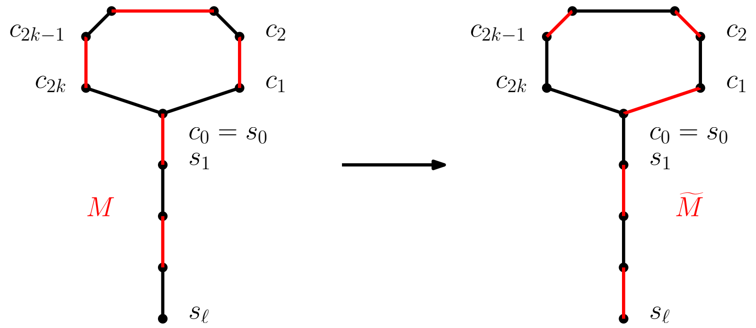
[7]Technically, we mean that G is obtained by contracting $V(C)$ to c.

Proof. The fact that M' is a matching of G' follows from the appropriate definitions.

Suppose first that M' is not a maximum matching of G'; we must show that M is not a maximum matching of G. Let M'' be a matching of G' of size greater than $|M'|$. If c_0 is unsaturated by M'', then $M'' \cup \Big(M \cap E(C)\Big)$ is a matching of G of size greater than $|M|$. Suppose now that c_0 is saturated by M''. Then there exists some vertex $x \in V(G) \setminus V(C)$ and an index $j \in \{0, \dots, 2k\}$ such that $xc_j \in E(G)$. But now the matching $(M'' \setminus \{xc_0\}) \cup \{xc_j\} \cup \{c_{j+1}c_{j+2}, c_{j+3}c_{j+4}, \dots, c_{j+2k-1}c_{j+2k}\}$ is a matching of G of size greater than $|M|$ (see the picture below).



Suppose now that M is not a maximum matching of G; we must show that M' is not a maximum matching of G'. First, let $\widetilde{M} := \Big(M \setminus (E(C) \cup E(S)\Big) \cup \{c_0c_1, c_2c_3, \dots, c_{2k-2}c_{2k-1}\} \cup \{s_1s_2, s_3s_4, \dots, s_{\ell-1}s_\ell\}$ and $\widetilde{M}' = \Big(M' \setminus E(S)\Big) \cup \{s_1s_2, s_3s_4, \dots, s_{\ell-1}s_\ell\}$.



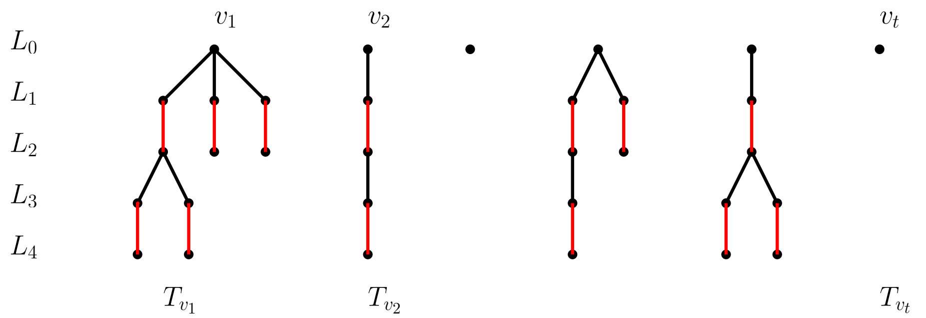
Clearly, $\widetilde{M}$ is a matching of G of the same size as M, and $\widetilde{M}'$ is a matching of G' of the same size as M'. Since the matching M of G is not maximum, neither is $\widetilde{M}$; so, by Theorem 1.2, there exists an $\widetilde{M}$-augmenting path in G, say $P = p_0, \dots, p_t$. It now suffices to exhibit an $\widetilde{M}'$-augmenting path in G', for Theorem 1.2 will then imply that the matching $\widetilde{M}'$ is not maximum in G', and consequently, M' is not maximum in G', either.

If $V(P) \cap V(C) = \emptyset$, then P is an $\widetilde{M}'$-augmenting path in G', and we are done. So, we may assume that $V(P) \cap V(C) \neq \emptyset$. First of all, c_{2k} is the only vertex in $V(C)$ that is unsaturated by $\widetilde{M}$; since both p_0, p_t are unsaturated by $\widetilde{M}$, we see that at most one of p_0, p_t belongs to $V(C)$. By symmetry, we may assume that $p_0 \notin V(C)$. Now, set $t_1 := \min\{i \in \{1, \dots, t\} \mid p_i \in V(C)\}$. But then $p_0, \ldots, p_{t_1-1}, c_0$ is an $\widetilde{M}'$-augmenting path in G',[8] and we are done. □

3 Edmonds' Blossom algorithm

Let G be an input graph. Initially, we start with the empty matching, and we iteratively increase the size of the matching until this is no longer possible, at which point, our matching is maximum. All we need to do is show how, given a matching M in G, we either produce a larger matching, or determine that no larger matching exists. We proceed as follows.

Step 1. First, we form an auxiliary forest F (which is a subgraph of G) as follows. $V(F)$ is partitioned into levels, $L_0, L_1, L_2, \ldots$. Level L_0 consists of all vertices of G that are unsaturated by M. Then, for each vertex $v \in L_0$, we use breadth-first-search to form a tree T_v rooted at v in which edges between levels alternate between edges of M and edges that are not in M.[9] L_k is the set of vertices at distance k (in F) from L_0. For an even k, edges between L_k and L_{k+1} in F do not belong to M, and edges between L_{k+1} and L_{k+2} in F do belong to M.[10]



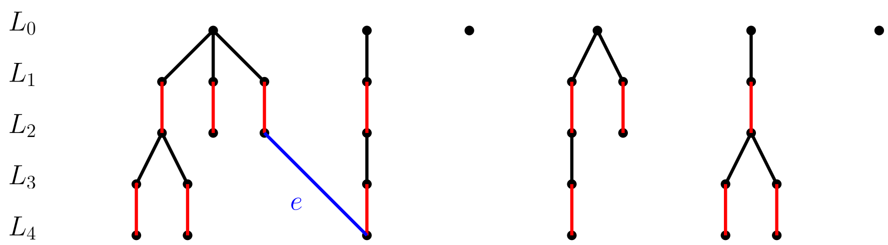
Step 2. If there exists an edge $e \in E(G)$ between even levels of two distinct trees, we immediately obtain an M-augmenting path,[11] and then we obtain a matching of size $|M| + 1$, as in Lemma 1.1.

[8] We are using the fact that, by construction, c_0 is unsaturated by $\widetilde{M}'$ in G'.

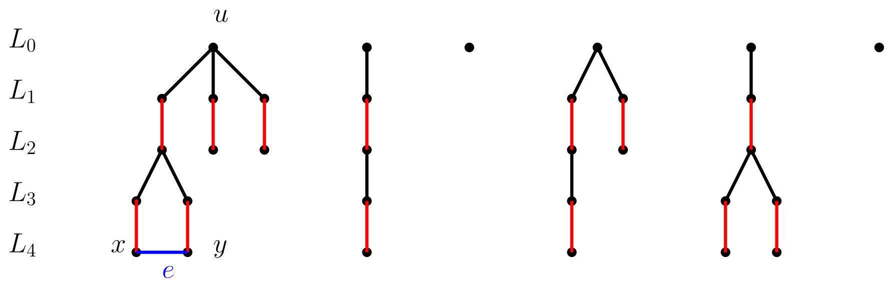
[9] If $L_0 = \{v_1, \ldots, v_t\}$, then we first form the tree T_{v_1} rooted at v_1. Having constructed trees $T_{v_1}, \ldots, T_{v_k}$ $(k \le t-1)$ rooted at $v_1, \ldots, v_k$, respectively, we perform the search starting at v_{k+1} in the graph $G \setminus (V(T_{v_1}) \cup \cdots \cup V(T_{v_k}))$, and we form the tree $T_{v_{k+1}}$ rooted at v_{k+1}.

[10] We remark that $V(F) \subseteq V(G)$, and it is possible that $V(F) \subsetneq V(G)$.

[11] Indeed, suppose that for distinct $u, v \in L_0$, and some even p, q, we have an edge e between a vertex $u' \in V(T_u) \cap L_p$ and a vertex $v' \in V(T_v) \cap L_q$. Let P_u be the unique path in T_u between u and u', and let P_v be the unique path in T_v between v and v'. Then $u - P_u - u' - v' - P_v - v$ is an M-augmenting path in G.



If there exists an edge $e \in E(G)$ between two vertices, say x and y, belonging to even levels of the same tree T_u,[12] then we can find a flower (i.e. a blossom with a corresponding stem), as follows.



We consider the (unique) path in T_u between x and u in T_u, and the (unique) path in T_u between y and u. The union of these two paths is a flower in G, say, with blossom $C = c_0, \ldots, c_{2k}, c_0$ and stem $S = s_0, \ldots, s_\ell$, where $c_0 = s_0$ and $s_\ell \in L_0$. Let G' be the graph obtained from G by contracting C to a vertex c_0, and let $M' = M \setminus E(C)$ (as in Lemma 2.1). We now call the algorithm with input G' and M'. Then there are two cases.

- If we obtain the answer that M' is a maximum matching in G', then (by Lemma 2.1) M is a maximum matching in G, and we are done.
- Suppose we obtained a matching M'' in G' that is of size greater than $|M'|$. If c_0 is unsaturated by M'', then $(E(C) \cap M) \cup M''$ is a matching in G of size greater than $|M|$, and we are done. Suppose now that c_0 is saturated by M'', and let $x \in V(G) \setminus V(C)$ be such that $xc_0 \in M''$. Let v be some vertex of C such that $xv \in E(G)$, and let M_C be the (unique) matching of size $\frac{|V(C)|-1}{2}$ in C, chosen so that v is M_C-unsaturated. Then $(M'' \setminus \{xc_0\}) \cup \{xv\} \cup M_C$ is a matching in G of size greater than $|M|$.

Suppose now that there are no edges (of G) between vertices in even levels. In this case, G contains no M-augmenting path,[13] and so by Theorem 1.2, M is a maximum matching of G.

[12] x and y may or may not belong to the same level. The important thing is that they are both in even levels, and they are in the same tree T_u.

[13] Let us check this. First, for a tree T and vertices $u, v \in V(T)$, we denote by $u - T - v$ the unique path between u and v in T. Now, suppose that there are no edges (of G)

Remark: The running time of Edmonds' Blossom algorithm is $O(n^4)$, if the algorithm is implemented in the obvious way. We omit the details.

---

between vertices in even levels of F, but that G nonetheless contains an M-augmenting path, say P, between some vertices $u, v \in L_0$. Then there exists a sequence $q_0, \dots, q_r$ of vertices in L_0, and for each $i \in \{0, \dots, r\}$, vertices $w_i, w_i' \in V(T_{q_i})$, such that $u = w_0$, $v = w_r'$, and $P = w_0 - T_{q_0} - w_0' - w_1 - T_{q_1} - w_1' - \cdots - w_r - T_{q_r} - w_r'$. Now, none of the edges $w_0'w_1, w_1'w_2, \dots, w_{r-1}'w_r$ is in M, which means that for all $i \in \{0, \dots, r-1\}$, the final edge of $w_i - T_{q_i} - w_i'$ is in M. In particular, $w_0', \dots, w_{r-1}'$ all belong to even levels. Since there are no edges between even levels, we see that $w_1, \dots, w_r$ all belong to odd levels. But then the first edge on the path $w_r - T_{q_r} - \underbrace{w_r'}_{=v}$ is an edge that does not belong to M. So, neither edge of the path P incident with w_r belongs to M, contrary to the fact that P is an M-augmenting path.