NDMI012: Combinatorics and Graph Theory 2
HW#8

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due Tuesday, May 18, 2021 before midnight (Prague time)

Remark: Please e-mail me (ipenev@iuuk.mff.cuni.cz) your HW as a **PDF attachment** (no other format will be accepted).

Problem 1 (40 points)**.** *Let p and ℓ be positive integers. Construct a family $\mathscr{A}$ of $(p-1)^\ell$ non-empty sets such that there does **not** exist a sunflower $\mathscr{S} \subseteq \mathscr{A}$ with p petals.*

Problem 2 (30 points)**.** *Prove that for all positive integers r and n, we have that*

$$t_r(n) \quad \leq \quad \tfrac{r-1}{2r}n^2,$$

and that equality holds whenever r divides n.

> ***Hint:*** *Set $n = kr + \ell$, where k and ℓ are non-negative integers with $\ell \leq r-1$. Treat the case $\ell = 0$ first, and then show for the general case that $t_r(n) = \frac{r-1}{2r}(n^2 - \ell^2) + \binom{\ell}{2}$.*

Problem 3 (30 points)**.** *Let r be a positive integer. Prove that*

$$\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} \quad = \quad \frac{r-1}{r}.$$

> ***Hint:*** *$r\lfloor \frac{n}{r} \rfloor \leq n \leq r\lceil \frac{n}{r} \rceil$, and consequently, $t_r\Big(r\lfloor \frac{n}{r} \rfloor\Big) \leq t_r(n) \leq t_r\Big(r\lceil \frac{n}{r} \rceil\Big)$. You may use Problem 2.*