NDMI012: Combinatorics and Graph Theory 2
HW#7

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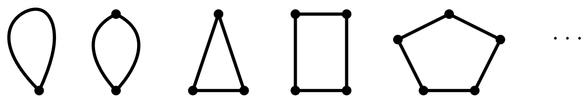
Winter 2020/2021

due Tuesday, May 11, 2021 before midnight (Prague time)

Remark: Please e-mail me (ipenev@iuuk.mff.cuni.cz) your HW as a **PDF attachment** (no other format will be accepted).

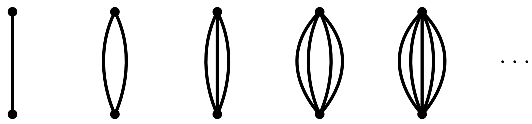
Problem 1 (20 points)**.**

(a) *[10 points] Prove that for all $n \geq 1$, $T_{C_n}(x, y) = y + \sum_{i=1}^{n-1} x^i$.*



C_1 C_2 C_3 C_4 C_5 $\dots$

(b) *[10 points] Prove that for $n \geq 1$, $T_{I_n}(x, y) = x + \sum_{i=1}^{n-1} y^i$, where I_n is the 2-vertex multigraph with n edges between them.*



I_1 I_2 I_3 I_4 I_5 $\dots$

Hint: *Induction on n, using the recursive deletion-contraction formula for the Tutte polynomial.*

Problem 2 (20 points)**.** *Prove that every cubic (i.e. 3-regular), Hamiltonian graph G satisfies $\chi'(G) = 3$.*

Problem 3 (20 points)**.** *Prove or disprove the following statement: "If G is a graph on at least three vertices, and G has at least $\alpha(G)$ universal vertices,*[1] *then G is Hamiltonian."*

> ***Remark:*** *So, if the statement is true, then you should prove it. If it false, then you should construct a counterexample (and prove that your counterexample really is a counterexample).*

Problem 4 (20 points)**.** *Let G be a graph that is not a forest and has girth at least five.*[2] *Prove that $\overline{G}$ (the complement of G) is Hamiltonian.*

> ***Hint:*** *Ore's theorem.*

Problem 5 (20 points)**.** *Prove that any non-Hamiltonian graph G on n vertices has at most $\binom{n-1}{2} + 1$ edges.*

> ***Hint:*** *Induction on n.*

[1] A *universal vertex* is a vertex that is adjacent to all other vertices of the graph.

[2] The *girth* of a graph that is not a forest is the length of the shortest cycle in the graph.