NDMI012: Combinatorics and Graph Theory 2
HW#6

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due Tuesday, April 20, 2021 before midnight (Prague time)

Remark: Please e-mail me (ipenev@iuuk.mff.cuni.cz) your HW as a **PDF attachment** (no other format will be accepted).

The Kőnig-Egerváry theorem. *The maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover in that graph.*

Problem 1 (40 points)**.** *Use Dilworth's theorem to prove the Kőnig-Egerváry theorem (stated above).*[1]

> ***Hint:*** *What is the most obvious way to "transform" a bipartite graph into a partially ordered set?*

Definition. *Given graphs H and K on disjoint vertex sets, and given a vertex $u \in V(H)$, we say that a graph G is obtained by* substituting *K for u in H provided the following hold:*

- $V(G) = (V(H) \setminus \{u\}) \cup V(K)$;
- $G[V(H) \setminus \{u\}] = H \setminus u$;
- $G[V(K)] = K$;
- *for all $u' \in V(K)$ and $v \in V(H) \setminus \{u\}$, u' is adjacent to v in G if and only if u is adjacent to v in H.*

[1] The Kőnig-Egerváry theorem was proven in Combinatorics & Graphs 1. Here, you are asked to give a different proof.

Problem 2 (60 points)**.** *In this problem, you may use* ***neither*** *the Perfect Graph Theorem* ***nor*** *the Strong Perfect Graph Theorem.*

(a) [30 points] Prove that the graph obtained by substituting a complete graph for a vertex of a perfect graph is perfect.[2]

> ***Hint:*** *Imitate the proof of the fact that α-perfection is preserved under vertex duplication. Think of a proper coloring as a partition into stable sets.*

(b) [30 points] Prove that the graph obtained by substituting a perfect graph for a vertex of a perfect graph is perfect.[3]

> ***Hint:*** *Suppose G is obtained by substituting K for a vertex u of H, where K and H are perfect graphs on disjoint vertex sets. Start by substituting a complete graph on $\omega(K)$ vertices for u in H, and use part (a). Now what?*

[2] In other words, prove that if H and K are graphs on disjoint vertex sets, H is perfect, K is complete, and $u \in V(H)$, then the graph G obtained by substituting K for u in H is perfect.

[3] In other words, prove that if H and K are perfect graphs on disjoint vertex sets and $u \in V(H)$, then the graph G obtained by substituting K for u in H is perfect.