NDMI011: Combinatorics and Graph Theory 1

Tutorial #8

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In what follows, our edge colorings need not be proper, i.e. it is possible that two edges that share an endpoint receive the same color.

Exercise 4 from Tutorial 7. *Prove that for every $k \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that for every graph G on at least n vertices, and every 2-edge-coloring of G, there exists some $U \subseteq V(G)$ such that $|U| = k$ and all the edges of $G[U]$ have the same color.*

Exercise 5 from Tutorial 7. *Let n be a non-negative integer, and let X be an n-element set. A* half-antichain *in $(\mathscr{P}(X), \subseteq)$ is a set $\mathcal{A}$ of subsets of X such that there do **not** exist sets $A_1, A_2, A_3 \in \mathcal{A}$ such that $A_1 \subsetneqq A_2 \subsetneqq A_3$.*

(a) Prove that any half-antichain in $(\mathscr{P}(X), \subseteq)$ has at most $2\binom{n}{\lfloor n/2 \rfloor}$ elements.

> ***Hint:*** *Imitate the proof of Sperner's theorem. You may use the statements of Claims 1 and 2 from the proof of Sperner's theorem without (re)proving them.*

(b) Prove that if n is odd, then there is a half-antichain in $(\mathscr{P}(X), \subseteq)$ that has precisely $2\binom{n}{\lfloor n/2 \rfloor}$ elements.

Exercise 1. *Let k be a positive integer. Using Ramsey numbers, prove that there exists a positive integer N such that any sequence $a_1, \dots, a_N$ of real numbers contains a subsequence of length k that is either strictly increasing, strictly decreasing, or constant.*

Schur's Theorem. *For any positive integer k, there exists a positive integer N such that for any coloring of the set $\{1, \dots, N\}$ with k colors, there exist $x, y, z \in \{1, \dots, N\}$ that have the same color and satisfy $x + y = z$.*

Exercise 2. *Prove Schur's theorem.*

Hint: *Take* $N = R^2(\underbrace{3, \ldots, 3}_{k})$*, and then color the edges of the complete graph* K_N *in a convenient way.*