NDMI011: Combinatorics and Graph Theory 1

Tutorial #7

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In what follows, our edge colorings need not be proper, i.e. it is possible that two edges that share an endpoint receive the same color.

Exercise 1. *Find the smallest $n \in \mathbb{N}$ such that, for every graph G on at least n vertices, either G contains K_3 as a subgraph, or $\overline{G}$ (the complement of G) contains $K_{1,3}$ as a subgraph.*

Exercise 2. *Let k and p be positive integers. Prove that there exists some positive integer n such that, whenever we color the edges of a complete graph on at least n vertices with p colors, the graph contains a monochromatic complete subgraph on at least k vertices.*[1]

> ***Hint:*** *Induction on p. For $p = 1$, this is trivial. For $p = 2$, this can easily be translated into Ramsey numbers (how?). Then what?*

Exercise 3. *Determine which of the following statements are true. If the statement is true, then prove it. If it is false, construct a counterexample. In what follows, k is a fixed positive integer.*

(a) *If we color a sufficiently large complete graph with a loop at each vertex (we color edges and loops) with 2 colors, we always obtain a monochromatic k-vertex complete subgraph with a loop at each vertex.*

(b) *There exists a positive integer n such that every graph G on at least n vertices contains either $K_{k,k}$ or $\overline{K_{k,k}}$ (the complement of $K_{k,k}$) as a subgraph.*

[1] "Monochromatic" means that all edges are colored with the same color.

(c) *There exists a positive integer n such that for every graph G on at least n vertices, G contains either $K_{k,k}$ or $\overline{K_{k,k}}$ as an induced subgraph.*

(d) *For every graph G, there exists a positive integer n such that in each 2-edge-coloring of K_n, there is an induced monochromatic copy of G.*[2]

Exercise 4. *Prove that for every $k \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that for every graph G on at least n vertices, and every 2-edge-coloring of G, there exists some $U \subseteq V(G)$ such that $|U| = k$ and all the edges of $G[U]$ have the same color.*

Exercise 5. *Let n be a non-negative integer, and let X be an n-element set. A* half-antichain *in $(\mathscr{P}(X), \subseteq)$ is a set $\mathcal{A}$ of subsets of X such that there do* ***not*** *exist sets $A_1, A_2, A_3 \in \mathcal{A}$ such that $A_1 \subsetneq A_2 \subsetneq A_3$.*

(a) *Prove that any half-antichain in $(\mathscr{P}(X), \subseteq)$ has at most $2\binom{n}{\lfloor n/2 \rfloor}$ elements.*

> ***Hint:*** *Imitate the proof of Sperner's theorem. You may use the statements of Claims 1 and 2 from the proof of Sperner's theorem without (re)proving them.*

(b) *Prove that if n is odd, then there is a half-antichain in $(\mathscr{P}(X), \subseteq)$ that has precisely $2\binom{n}{\lfloor n/2 \rfloor}$ elements.*

[2] So, the edges of G are colored with one color, and the non-edges of G with the other color.