NDMI011: Combinatorics and Graph Theory 1

Tutorial #4

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Exercise 5 from Tutorial 3. *Using Tutte's theorem, prove the "(a) $\Longrightarrow$ (b)" part of the graph theoretic formulation of Hall's theorem. More precisely, let G be a bipartite graph with bipartition (A, B), and assume that all sets $A' \subseteq A$ satisfy $|A'| \leq |N_G(A')|$. Using Tutte's theorem, prove that G has an A-saturating matching.*

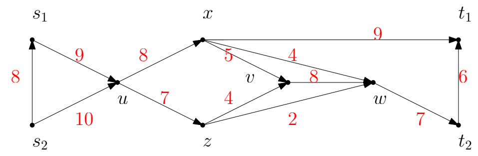
> ***Hint:*** *First, reduce the problem to the case when $|A| = |B|$. (If $|A| \neq |B|$, then add vertices and edges to G in a convenient way.) Then, show that for each $S \subsetneq V(G)$, the number of components C of $G \setminus S$ such that $|A \cap V(C)| \neq |B \cap V(C)|$ is at most $|S|$. Now what?*

Exercise 1. *Let (G, s, t, c) be a network. Prove that some maximum flow f in (G, s, t, c) satisfies the property that the in-flow into s and the out-flow from t are both zero, i.e. $\sum\limits_{(x,s) \in E(G)} f(x, s) = \sum\limits_{(t,x) \in E(G)} f(t, x) = 0$.*

> ***Hint:*** *Let f be a maximum flow for which the number of edges $(x, y) \in E(G)$ such that $f(x, y) = 0$ is as large as possible. Now show that f has the desired property.*

Exercise 2. *Explain how the problem of finding a maximum flow in a network with more than one source or more than one sink can be reduced to the usual problem of finding a maximum flow in a network with one source and one sink. (You do not have to give a formal proof of correctness; just explain the construction.[1]) Then, find a maximum flow in the network below, where s_1, s_2 are sources and t_1, t_2 are sinks.*

[1] Or, if you want a bit of a challenge, try to prove the correctness of your construction.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