NDMI011: Combinatorics and Graph Theory 1

Tutorial #3

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Exercise 1. *Let X be the set of points and $\mathcal{P}$ the set of lines constructed in the proof of Theorem 2.3 from Lecture Notes 5. Using Exercise 2 from Tutorial 2 (or in some other way),*[1] *show that $(X, \mathcal{P})$ is indeed a finite projective plane of order n. Make sure you carefully indicate where you are using the fact that $L_1, \dots, L_{n-1}$ are Latin squares, and where you are using orthogonality.*

Exercise 2.

(a) Give an example of a non-bipartite graph G_1 in which the maximum size of a matching is strictly smaller than the minimum size of a vertex cover.

(b) Give an example of a non-bipartite graph G_2 in which the maximum size of a matching is equal to the minimum size of a vertex cover.

In both parts, make sure you prove that your answer is correct.

> ***Remark:*** *Recall that a graph is bipartite if and only if it contains no odd cycles. So, make sure that both G_1 and G_2 have an odd cycle. Try to make your G_1 and G_2 as small as possible (there are some very small examples.)*

[1] This exercise (or rather, just part (e)) is stated below. You are supposed to **use** the statement from the exercise below, not prove it.

Exercise 2 from Tutorial 2. *Let $n \geq 2$ be an integer, and let $(X, \mathcal{P})$ be a set system*[2] *such that $|X| = |\mathcal{P}| = n^2 + n + 1$. Assume that $|P| = n + 1$ for all $P \in \mathcal{P}$, and assume furthermore that all distinct $P_1, P_2 \in \mathcal{P}$ satisfy $|P_1 \cap P_2| \leq 1$. In what follows, we refer to the elements of X as* points *and to the elements of $\mathcal{P}$ as* lines.

(e) Prove that $(X, \mathcal{P})$ is a finite projective plane of order n.

Exercise 3. *Let k be a positive integer. Prove that $\lambda(K_{k+1}) = k$.*

> ***Remark:*** *This result was used in the proof of Theorem 3.3 from Lecture Notes 7.*

Exercise 4. *Using the graph theoretic formulation of Hall's theorem (the one with matchings), prove the combinatorial formulation of Hall's theorem (the one with transversals).*

Exercise 5. *Using Tutte's theorem, prove the "(a) $\Longrightarrow$ (b)" part of the graph theoretic formulation of Hall's theorem. More precisely, let G be a bipartite graph with bipartition (A, B), and assume that all sets $A' \subseteq A$ satisfy $|A'| \leq |N_G(A')|$. Using Tutte's theorem, prove that G has an A-saturating matching.*

> ***Hint:*** *First, reduce the problem to the case when $|A| = |B|$. (If $|A| \neq |B|$, then add vertices and edges to G in a convenient way.) Then, show that for each $S \subsetneq V(G)$, the number of components C of $G \setminus S$ such that $|A \cap V(C)| \neq |B \cap V(C)|$ is at most $|S|$. Now what?*