

# NDMI011: Combinatorics and Graph Theory 1

## Tutorial #2

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**Exercise 1.** Let  $(X, \mathcal{P})$  be a set system such that  $X$  is finite, and all the following are satisfied:<sup>1</sup>

(P0') there exist distinct  $P_1, P_2 \in \mathcal{P}$  such that  $|P_1|, |P_2| \geq 3$ ;

(P1) all distinct  $P_1, P_2 \in \mathcal{P}$  satisfy  $|P_1 \cap P_2| = 1$ ;

(P2) for all distinct  $x_1, x_2 \in X$ , there exists a unique  $P \in \mathcal{P}$  such that  $x_1, x_2 \in P$ .

Prove that  $(X, \mathcal{P})$  is a finite projective plane.

**Exercise 2.** Let  $n \geq 2$  be an integer, and let  $(X, \mathcal{P})$  be a set system<sup>2</sup> such that  $|X| = |\mathcal{P}| = n^2 + n + 1$ . Assume that  $|P| = n + 1$  for all  $P \in \mathcal{P}$ , and assume furthermore that all distinct  $P_1, P_2 \in \mathcal{P}$  satisfy  $|P_1 \cap P_2| \leq 1$ . In what follows, we refer to the elements of  $X$  as points and to the elements of  $\mathcal{P}$  as lines.

(a) Prove that for all distinct points  $x_1, x_2 \in X$ , there exists a unique line  $P \in \mathcal{P}$  such that  $x_1, x_2 \in P$ .

**Hint:** Uniqueness should be easy (details?). For existence, start by noticing that for each line  $P$ , there are  $\binom{n+1}{2}$  unordered pairs of distinct points of  $P$ . And how many unordered pairs of distinct points of  $X$  are there? Now what?

(b) Prove that each point is contained in at most  $n + 1$  lines.

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<sup>1</sup>Note that (P1) and (P2) are exactly as in the definition of a finite projective plane, but we have (P0') instead of (P0).

<sup>2</sup>So,  $X$  is a set and  $\mathcal{P} \subseteq \mathcal{P}(X)$ , where  $\mathcal{P}(X)$  is the power set (i.e. the set of all subsets) of  $X$ .

**Hint:** Show that otherwise, you get “too many” points in  $X$ .

(c) Prove that each point is contained in precisely  $n + 1$  lines.

**Hint:** Consider the incidence graph of the set system  $(X, \mathcal{P})$ , and then count the number of edges in two ways: first using the degrees of points, and then using the degrees of lines. Use part (b).

(d) Prove that every two lines in  $\mathcal{P}$  intersect, that is, prove that for all  $P_1, P_2 \in \mathcal{P}$ , we have that  $P_1 \cap P_2 \neq \emptyset$ .

**Hint:** Fix an arbitrary line  $P \in \mathcal{P}$ . How many lines is it intersected by? (You should use one of the previous parts here.)

(e) Prove that  $(X, \mathcal{P})$  is a finite projective plane of order  $n$ .

**Hint:** You might want to use Exercise 1, plus the previous parts.