

NDMI011: Combinatorics and Graph Theory 1

Lecture #11

Ramsey theory and König's infinity lemma

Irena Penev

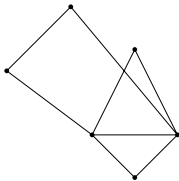
December 15, 2021

- Reminder: For positive integers k and ℓ , $R(k, \ell)$ the smallest $N \in \mathbb{N}$ such that every graph G on at least N vertices satisfies either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

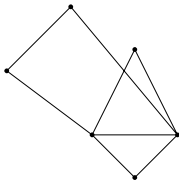
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- Numbers $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) are called *Ramsey numbers*, and we proved that they exist in Lecture Notes 10.
- There's another way to think about Ramsey numbers!

- Any graph G corresponds to a complete graph on the same vertex set, and whose edges are colored black or white, with an edge colored black if it was an edge of the graph G , and colored white otherwise.

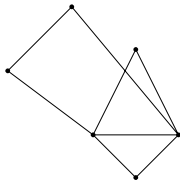


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- Now $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) is the smallest $N \in \mathbb{N}$ such that any complete graph on at least N vertices, and whose edges are colored black or white, has either a monochromatic black complete subgraph of size k , or a monochromatic white complete subgraph of size ℓ .

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- Now $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) is the smallest $N \in \mathbb{N}$ such that any complete graph on at least N vertices, and whose edges are colored black or white, has either a monochromatic black complete subgraph of size k , or a monochromatic white complete subgraph of size ℓ .
- If instead of black and white, we use colors 1 and 2, then a coloring of the complete graph on vertex set X is simply a function $c : \binom{X}{2} \rightarrow [2]$.
 - $\binom{X}{p}$ is the set of all p -element subsets of X .

- So, $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) is the smallest $N \in \mathbb{N}$ such that for all finite sets X with $|X| \geq N$, and all colorings $c : \binom{X}{2} \rightarrow [2]$, either there exists a set $A_1 \in \binom{X}{k}$ such that c assigns color 1 to each set in $\binom{A_1}{2}$, or there exists a set $A_2 \in \binom{X}{\ell}$ such that c assigns color 2 to each set in $\binom{A_2}{2}$.

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- This can be generalized!

Definition

A *hypergraph* is an ordered pair $H = (V(H), E(H))$, where $V(H)$ is some non-empty finite set, and $E(H) \subseteq \mathcal{P}(V(H)) \setminus \{\emptyset\}$. Members of $V(H)$ are called *vertices* and members of $E(H)$ are called *edges* of the hypergraph H .

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- So, if H is a p -uniform hypergraph, then $E(H) \subseteq \binom{V(H)}{p}$.
- A graph is simply a 2-uniform hypergraph.

Definition

Given $p, t, k_1, \dots, k_t \in \mathbb{N}$, the *Ramsey number* $R^p(k_1, \dots, k_t)$ is the smallest $N \in \mathbb{N}$ (if it exists) such that for all finite sets X with $|X| \geq N$, and all colorings (i.e. functions) $c : \binom{X}{p} \rightarrow [t]$,^a there exist an index $i \in [t]$ and a set $A_i \in \binom{X}{k_i}$ such that c assigns color i to each element of $\binom{A_i}{p}$.

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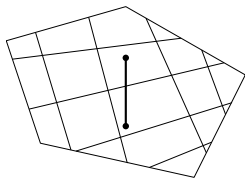
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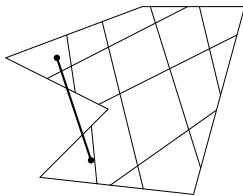
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- Here, we will present only the second proof.
- But first, let's look at a geometric application!

Definition

A set X of points in the plane is *convex* if for all distinct $x_1, x_2 \in X$, the line segment between x_1 and x_2 lies in X . The *convex hull* of a non-empty set S of points in the plane is the smallest convex set in the plane that includes S .

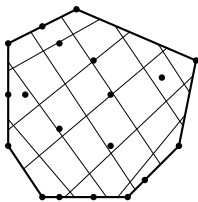


convex

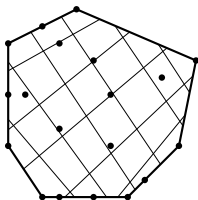


non-convex

- If S is a finite set of points in the plane containing at least three non-collinear points, then the convex hull of S is a convex polygon (with its interior), and the vertices of this polygon are all in S .



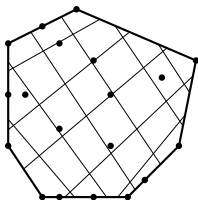
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(Pairwise distinct) points x_1, \dots, x_t ($t \geq 3$) in the plane are in *convex position* if they are the vertices of some convex polygon.

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- Equivalently, (pairwise distinct) points x_1, \dots, x_t ($t \geq 3$) in the plane are in convex position if their convex hull is a convex t -gon whose vertices are precisely x_1, \dots, x_t (not necessarily in that order).

Lemma 1.1

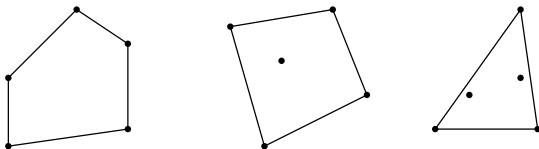
Any set of five points in the plane, no three of which are collinear, contains four points in convex position.

Proof (outline).

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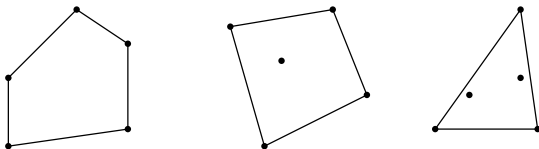
Proof (outline). Let a_1, \dots, a_5 be five points in the plane, no three of which are collinear. We now consider the convex hull of these five points.



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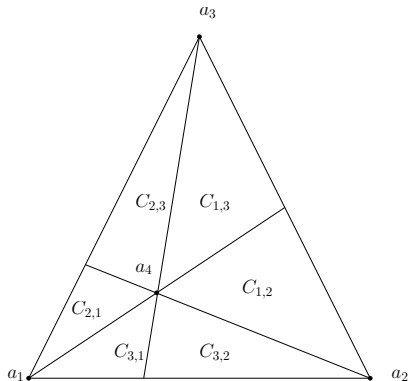


WMA the convex hull is a triangle, for otherwise we are done.

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Proof (outline, continued).



If $a_5 \in C_{i,j}$, then a_i, a_4, a_5, a_j are the vertices of a convex quadrilateral, and we are done.

The Erdős-Szekeres theorem

Let $t \geq 4$ be an integer. Any set of at least $R^4(5, t)$ points in the plane, no three of which are collinear, contains t points in convex position.

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Proof (outline, continued). Suppose that there exists some $A_1 \in \binom{S}{5}$ such that c assigns color 1 to all elements of $\binom{A_1}{4}$.

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Let $t \geq 4$ be an integer. Any set of at least $R^4(5, t)$ points in the plane, no three of which are collinear, contains t points in convex position.

Proof (outline, continued). Suppose that there exists some $A_1 \in \binom{S}{5}$ such that c assigns color 1 to all elements of $\binom{A_1}{4}$. Then A_1 is a set of five points in the plane, no three of which are collinear, and no four of which are in convex position. But this contradicts Lemma 1.1.

The Erdős-Szekeres theorem

Let $t \geq 4$ be an integer. Any set of at least $R^4(5, t)$ points in the plane, no three of which are collinear, contains t points in convex position.

Proof (outline, continued).

The Erdős-Szekeres theorem

Let $t \geq 4$ be an integer. Any set of at least $R^4(5, t)$ points in the plane, no three of which are collinear, contains t points in convex position.

Proof (outline, continued). It now follows that there exists some $A_2 \in \binom{S}{t}$ such that c assigns color 2 to all elements of $\binom{A_2}{4}$. Then A_2 is a set of t points in the plane, no three of which are collinear, and any four of which are in convex position.

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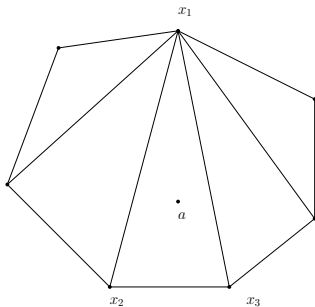
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Proof (outline, continued).



If some point of S is not a vertex of the polygon, then we get four points of A_4 that are not in convex position.

Ramsey's theorem (infinite version)

For all $t, p \in \mathbb{N}$, all infinite sets X , and all colorings $c : \binom{X}{p} \rightarrow [t]$, there exists an infinite set $A \subseteq X$ such that $c \upharpoonright \binom{A}{p}$ is constant.

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Proof. We fix $t \in \mathbb{N}$, and we proceed by induction on p .

For $p = 1$, we fix an infinite set X and a coloring $c : \binom{X}{1} \rightarrow [t]$. For all $i \in [t]$, we set $C_i = \{x \in X \mid c(\{x\}) = i\}$. Then (C_1, \dots, C_t) is a partition of X , and consequently, at least one of the sets C_1, \dots, C_t , say C_i , is infinite. Furthermore, $c \upharpoonright \binom{C_i}{1}$ is constant (indeed, it assigns color i to each element of $\binom{C_i}{1}$). So, the theorem is true for $p = 1$.

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Proof (continued). Fix $p \in \mathbb{N}$, and assume the theorem is true for p . We must show that it is true for $p + 1$.

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Proof (continued). Fix $p \in \mathbb{N}$, and assume the theorem is true for p . We must show that it is true for $p + 1$. Fix an infinite set X and a coloring $c : \binom{X}{p+1} \rightarrow [t]$. Our goal is to recursively construct a sequence $\{X_n\}_{n=1}^{\infty}$ of infinite subsets of X and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X with the following three properties:

- $x_n \in X_n$ for all $n \in \mathbb{N}$;
- $X_{n+1} \subseteq X_n \setminus \{x_n\}$ for all $n \in \mathbb{N}$;
- for all $n \in \mathbb{N}$, c assigns the same color to all sets of the form $\{x_n\} \cup Y$, with $Y \in \binom{X_{n+1}}{p}$.

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Proof (continued). First, we set $X_1 = X$ and we choose $x_1 \in X$ arbitrarily. Now, having constructed X_1, \dots, X_n and x_1, \dots, x_n , we construct X_{n+1} and x_{n+1} as follows. We define an auxiliary coloring $c_n : \binom{X_n \setminus \{x_n\}}{p} \rightarrow [t]$ by setting $c_n(A) = c(A \cup \{x_n\})$ for all $A \in \binom{X_n \setminus \{x_n\}}{p}$. Since $X_n \setminus \{x_n\}$ is infinite, the induction hypothesis guarantees that there exists some infinite set $X_{n+1} \subseteq X_n \setminus \{x_n\}$ such that $c_n \upharpoonright \binom{X_{n+1}}{p}$ is constant. But now by construction, we have that c assigns the same color to all sets of the form $\{x_n\} \cup Y$, with $Y \in \binom{X_{n+1}}{p}$. Finally, we choose $x_{n+1} \in X_{n+1}$ arbitrarily.

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Proof (continued). We have now constructed a sequence $\{X_n\}_{n=1}^{\infty}$ of infinite subsets of X and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X with the following three properties:

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It follows from the construction that for all $n \in \mathbb{N}$, the coloring c assigns the same color to all sets of the form $\{x_n\} \cup \{x_{j_1}, \dots, x_{j_p}\}$, with $n < j_1 < \dots < j_p$; let us say this color is *associated* with x_n .

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Proof (continued). Reminder: For all $n \in \mathbb{N}$, the coloring c assigns the same color to all sets of the form $\{x_n\} \cup \{x_{j_1}, \dots, x_{j_p}\}$, with $n < j_1 < \dots < j_p$; this color is *associated* with x_n .

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For all $i \in [t]$, we let $A_i = \{x_n \mid n \in \mathbb{N}, i \text{ is associated with } x_n\}$. Then (A_1, \dots, A_t) is a partition of the infinite set $\{x_1, x_2, x_3, \dots\}$, and we deduce that at least one of the sets A_1, \dots, A_t , say A_i , is infinite. But now $c \upharpoonright \binom{A_i}{p+1}$ is constant (it assigns i to all elements of $\binom{A_i}{p+1}$). This completes the induction.

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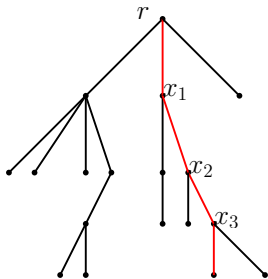
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Definition

Given $p, t, k_1, \dots, k_t \in \mathbb{N}$, the *Ramsey number* $R^p(k_1, \dots, k_t)$ is the smallest $N \in \mathbb{N}$ (if it exists) such that for all finite sets X with $|X| \geq N$, and all colorings (i.e. functions) $c : \binom{X}{p} \rightarrow [t]$,^a there exist an index $i \in [t]$ and a set $A_i \in \binom{X}{k_i}$ such that c assigns color i to each element of $\binom{A_i}{p}$.

^aSo, c is an assignment of colors to the edges of the “complete” p -uniform hypergraph on vertex set X .

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$p, t, k \in \mathbb{N}$, the number $R^p(\underbrace{k, \dots, k}_t)$ does not exist. Now, for each

integer $n \geq p$, we say that a coloring $c : \binom{[n]}{p} \rightarrow [t]$ is *n-bad* if

there is no set $A \in \binom{[n]}{k}$ such that $c \upharpoonright \binom{A}{p}$ is constant; a coloring is *bad* if it is *n-bad* for some integer $n \geq p$.

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there is no set $A \in \binom{[n]}{k}$ such that $c \upharpoonright \binom{A}{p}$ is constant; a coloring is *bad* if it is *n-bad* for some integer $n \geq p$. Since $R^p(\underbrace{k, \dots, k}_t)$ does

not exist, we see that for all integers $n \geq p$, there is at least one *n-bad* coloring.

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Proof (continued). Now, let C be the set of all bad colorings, and let T be the graph on the vertex set $C \cup \{r\}$ (where $r \notin C$), with adjacency as follows:

- r is adjacent to all p -bad colorings, and to no other elements of C ;
- for all integers $n \geq p$, n -bad colorings are pairwise non-adjacent;
- for all integers $n \geq p$, an n -bad coloring c_n is adjacent to an $(n+1)$ -bad coloring c_{n+1} iff c_{n+1} is an extension of c_n ;
- for all integers $n_1, n_2 \geq p$ such that $|n_1 - n_2| \geq 2$, no n_1 -bad coloring is adjacent to any n_2 -bad coloring.

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Now (T, r) is a rooted tree. Furthermore, for each integer $n \geq p$, the number of n -bad colorings is finite, and it follows from the construction of T that the T is locally finite.

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Now (T, r) is a rooted tree. Furthermore, for each integer $n \geq p$, the number of n -bad colorings is finite, and it follows from the construction of T that the T is locally finite. So, by König's infinity lemma, there is a ray $r, c_p, c_{p+1}, c_{p+2}, \dots$ in T .

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