NDMI011: Combinatorics and Graph Theory 1

Lecture #9

Triangle-free graphs and graphs without a C_4 subgraph. Cayley's formula

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December 1, 2021

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1. bounding the number of edges in graphs without certain subgraphs;

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2. the number of spanning trees of K_n (Cayley's formula).

Part I: Bounding the number of edges in graphs without certain subgraphs

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- Remark: All bipartite graphs are triangle-free.
- However, there are plenty of triangle-free graphs that are not bipartite! (Example: odd cycles of length ≥ 5.)

Mantel's theorem

Let n be a positive integer. Then

(a) any triangle-free graph on n vertices has at most $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$ edges;

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For (b), we observe that the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is triangle-free and has precisely n vertices and $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ edges.

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Assume inductively that (a) holds for all $n' < n$; we must prove it for n. For $n = 1$ and $n = 2$, this is obvious. So, suppose $n \geq 3$, and let G be a triangle-free graph on n vertices.

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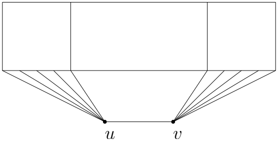
Proof (outline, continued). WMA G has at least one edge (say, uv), for otherwise we are done. No vertex in $V(G) \setminus \{u, v\}$ is adjacent to both u and v (otherwise, we'd get a triangle).



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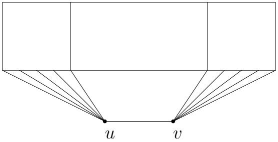
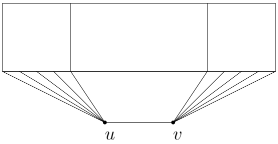
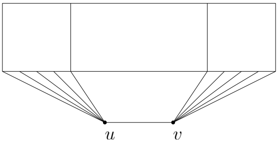


By the induction hypothesis, $|E(G \setminus \{u, v\})| \leq \lfloor \frac{(n-2)^2}{4} \rfloor$.



Proof (outline, continued). Reminder: $|E(G \setminus \{u, v\})| \leq \lfloor \frac{(n-2)^2}{4} \rfloor$.



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Since the edges of G are precisely the edges of $G \setminus \{u, v\}$, plus the edges between $\{u, v\}$ and $V(G) \setminus \{u, v\}$, plus the edge uv, we see that

$$\begin{aligned} |E(G)| &\leq \lfloor \tfrac{(n-2)^2}{4} \rfloor + (n-2) + 1 \\ &= \lfloor \tfrac{n^2-4n+4}{4} \rfloor + n - 1 \\ &= \lfloor \tfrac{n^2}{4} \rfloor, \end{aligned}$$

which is what we needed to show.

The Cauchy-Schwarz inequality

All real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ satisfy

$$\Big(\sum_{i=1}^n a_i b_i\Big)^2 \;\le\; \Big(\sum_{i=1}^n a_i^2\Big)\Big(\sum_{i=1}^n b_i^2\Big).$$

Proof. Omitted.

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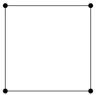
Let $n \in \mathbb{N}$. Any graph on n vertices that does not contain C_4 as a subgraph has at most $\frac{1}{2}(n + n^{3/2})$ edges.



C_4

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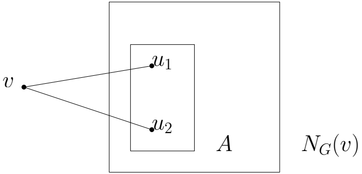
Let $M = \{(v, A) \mid v \in V(G), A \in \binom{N_G(v)}{2}\}$.



Now, we will count the number of elements of M in two ways.

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First, for each $v \in V(G)$, there are precisely $\binom{d_G(v)}{2}$ choices of A such that $(v, A) \in M$. So, $|M| = \sum\limits_{v \in V(G)} \binom{d_G(v)}{2} = \sum\limits_{i=1}^{n} \binom{d_i}{2}$.

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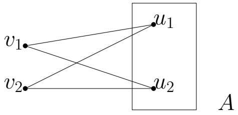
We now bound $|M|$ above, as follows. Since G contains no C_4 as a subgraph, we see that no two distinct elements of M have the same second coordinate. So, $|M| \leq \binom{n}{2}$.



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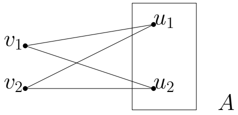

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Obviously, $\binom{n}{2} \le \frac{n^2}{2}$, and since $d_1, \ldots, d_n \ge 1$, we see that $\binom{d_i}{2} \ge \frac{(d_i-1)^2}{2}$ for all $i \in \{1, \ldots, n\}$; consequently,

$$\sum_{i=1}^{n} \frac{(d_i-1)^2}{2} \quad \le \quad \sum_{i=1}^{n} \binom{d_i}{2} \quad \le \quad \binom{n}{2} \quad \le \quad \frac{n^2}{2},$$

and it follows that

$$\sum_{i=1}^{n} (d_i - 1)^2 \quad \le \quad n^2.$$

- Cauchy-Schwarz: $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$.

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$$\begin{aligned}
\sum_{i=1}^{n}(d_i - 1) &= \sum_{i=1}^{n}(d_i - 1)\cdot 1 \\
&\le \sqrt{\sum_{i=1}^{n}(d_i - 1)^2}\sqrt{\sum_{i=1}^{n} 1^2} && \text{by C-S} \le \\
&= \sqrt{\sum_{i=1}^{n}(d_i - 1)^2}\sqrt{n} \\
&\le \sqrt{n^2}\sqrt{n} && \text{because } \sum_{i=1}^{n}(d_i - 1)^2 \le n^2 \\
&= n^{3/2}.
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It now follows that

$$|E(G)| \;=\; \frac{1}{2}\sum_{i=1}^{n} d_i \;=\; \frac{1}{2}\Big(n + \sum_{i=1}^{n}(d_i - 1)\Big) \;\leq\; \frac{1}{2}(n + n^{3/2}),$$

which is what we needed to show.

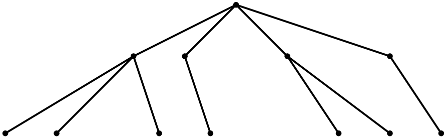
Part II: Cayley's formula

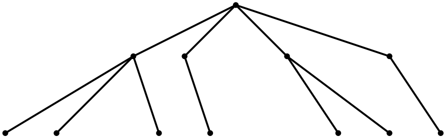
Definition

A *forest* is an acyclic graph (i.e. a graph that has no cycles), and a *tree* is a connected forest.

Definition

A *leaf* in a graph G is a vertex of degree one, i.e. a vertex that has exactly one neighbor.





Fact

Every tree on at least two vertices has at least two leaves.

Fact

If v is a leaf of a tree T, then $T \setminus v$ is a tree.

Definition

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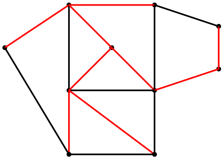

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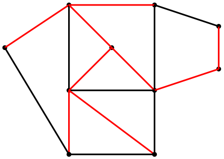

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- In other words, we would like to count the number of trees on the vertex set $\{1, \ldots, n\}$.

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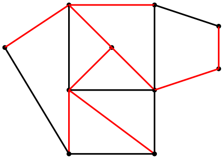

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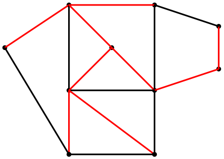

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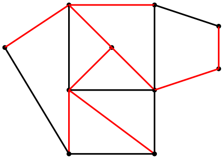

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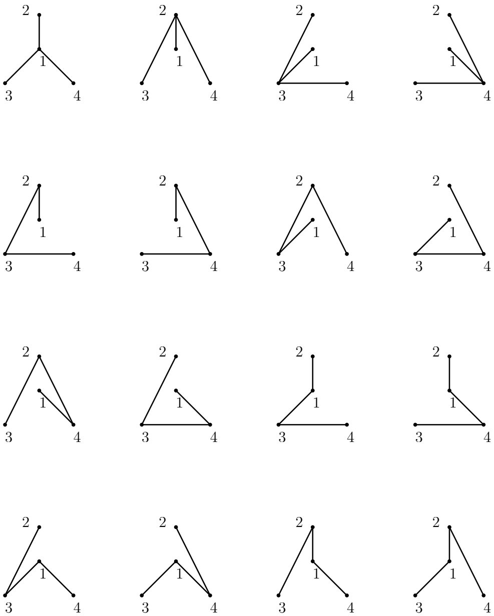
- For $n = 2$, there is one such tree.
- For $n = 3$, there are three such trees.
- For $n = 4$, there are 16 such trees.



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Lemma 3.4

Let $n \geq 2$ be an integer, and let $S \subseteq \mathbb{N}$ be such that $|S| = n$. Then the number of trees on the vertex set S is n^{n-2}.

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    - However, this is **not** standard terminology. We simply use it as a convenient shorthand in this lecture.

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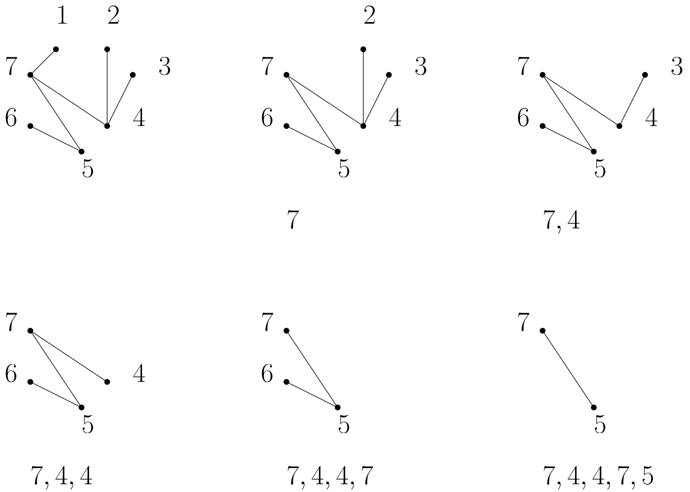
Definition

We define the *Prüfer code* of integer trees on at least two vertices recursively, as follows:

- for any integer tree T on exactly two vertices, the Prüfer code of T, denoted by $P(T)$, is the empty sequence;
- for any integer tree T on at least three vertices, we define the Prüfer code of T to be $P(T) := a_i, P(T \setminus i)$, where i is the smallest leaf of T, and a_i is the unique neighbor of i in T.[a]

[a] So, $P(T)$ is obtained by adding a_i to the front of $P(T \setminus i)$.

- For example, the Prüfer code of the tree in the top left corner is $7, 4, 4, 7, 5$, as shown below:



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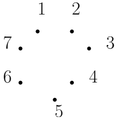
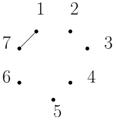
- For an integer $n \geq 2$, an n-element set $S \subseteq \mathbb{N}$, and an $(n-2)$-term sequence P, with terms in S, we proceed as follows.
  1. If $n \geq 3$, then we let i be the smallest element of S that is not in P, and we let a_i be the first term of P. We make i and a_i adjacent, we delete i from S, and we delete the first term of P.
  2. We repeat the process until S only has two elements left, and P is the empty sequence. At this point, we make the last two remaining elements of S adjacent.

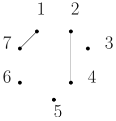
- For example, the tree on the vertex set $S = \{1, 2, 3, 4, 5, 6, 7\}$ whose Prüfer code is $7, 4, 4, 7, 5$ is the tree on the bottom of the picture (e is the empty sequence).



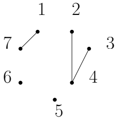
$P = 7, 4, 4, 7, 5$
$S = \{1, 2, 3, 4, 5, 6, 7\}$



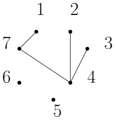
$P = 4, 4, 7, 5$
$S = \{2, 3, 4, 5, 6, 7\}$



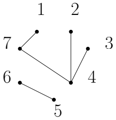
$P = 4, 7, 5$
$S = \{3, 4, 5, 6, 7\}$



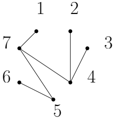
$P = 7, 5$
$S = \{4, 5, 6, 7\}$



$P = 5$
$S = \{5, 6, 7\}$



$P = e$
$S = \{5, 7\}$



Lemma 3.4

Let $n \geq 2$ be an integer, and let $S \subseteq \mathbb{N}$ be such that $|S| = n$. Then the number of trees on the vertex set S is n^{n-2}.

Proof (outline).

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Proof (outline). The mapping $T \mapsto P(T)$ is a bijection from the set of all integer trees on the vertex set S to the set of $(n-2)$-term sequences, all of whose terms are elements of S (details: Lecture Notes).

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Cayley's formula

For all $n \geq 2$, the number of spanning trees of K_n is n^{n-2}.

Proof. This follows immediately from Lemma 3.4, for $S = \{1, \ldots, n\}$.

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- One proof uses the "Laplacians" (matrices).
- In fact, one can use the "Laplacian" of an arbitrary graph (on vertex set $\{1, \ldots, n\}$) to compute the number of spanning trees of that graph.
- We give the formula without proof.

Definition

Suppose that $n \geq 2$ is an integer, and that G is a graph on the vertex set $\{1, \ldots, n\}$. Then the *Laplacian* of G is the matrix $Q = [q_{i,j}]_{n \times n}$ given by

$$q_{i,j} = \begin{cases} d_G(i) & \text{if} \quad i = j \\ -1 & \text{if} \quad i \neq j \text{ and } ij \in E(G) \\ 0 & \text{if} \quad i \neq j \text{ and } ij \notin E(G) \end{cases}$$

Theorem 3.5

Let $n \geq 2$ be an integer, let G be any graph on the vertex set $\{1, \ldots, n\}$, and let Q be the Laplacian of G. Then the number of spanning trees of G is precisely $\det(Q_{1,1})$.[a]

[a] $Q_{1,1}$ is the matrix obtained from Q by deleting the first row and first column.

Example

Using Theorem 3.5, prove Cayley's formula.

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Solution. Fix an integer $n \geq 2$, and consider the complete graph on the vertex set $\{1, \ldots, n\}$. Then the Laplacian of this graph is the $n \times n$ matrix

$$Q = \begin{bmatrix} n-1 & -1 & -1 & \ldots & -1 \\ -1 & n-1 & -1 & \ldots & -1 \\ -1 & -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \ldots & n-1 \end{bmatrix}_{n \times n}.$$

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The matrix $Q_{1,1}$ has exactly the same form, only it is of size $(n-1) \times (n-1)$. Since $\det(Q_{1,1}) = n^{n-2}$ (details: Lecture Notes), Theorem 3.5 guarantees that the number of spanning trees of K_n is n^{n-2}. This proves Cayley's formula.