NDMI011: Combinatorics and Graph Theory 1

Lecture #8

Menger's theorems and the Ear lemma

Irena Penev

November 24, 2021

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- This lecture consists of three parts:
  1. a brief review of vertex- and edge-connectivity;
  2. Menger's theorems;
  3. 2-connected graphs and the Ear lemma.

Part I: A brief review of vertex- and edge-connectivity

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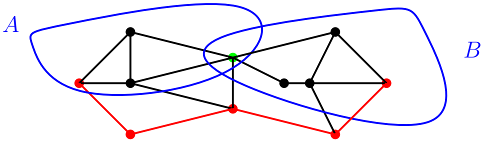
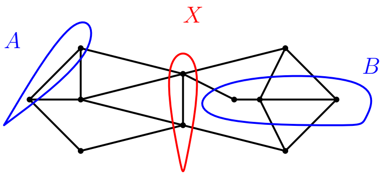
Definition

For a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, an *A-B path* in G, or a *path from A to B* in G, is either a one-vertex path whose sole vertex is in $A \cap B$, or a path on at least two vertices whose one endpoint is in A and whose other endpoint is in B.



Definition

Given a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, we say that a set $X \subseteq V(G)$ *separates* A from B in G if every path from A to B in G contains at least one vertex of X. Note that this implies that $A \cap B \subseteq X$.



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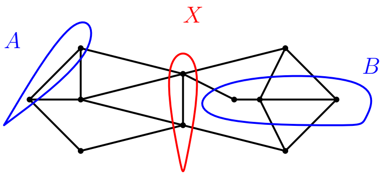
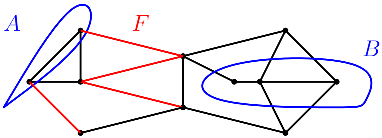


Definition

Given a graph G and a non-negative integer k, we say that G is *k-vertex-connected*, or simply *k-connected*, if $|V(G)| \geq k + 1$ and for all $X \subseteq V(G)$ s.t. $|X| \leq k - 1$, we have that $G \setminus X$ is connected.

Definition

Given a graph G and disjoint sets $A, B \subseteq V(G)$, we say that a set $F \subseteq E(G)$ *separates* A from B in G if every path from A to B contains at least one edge of F.



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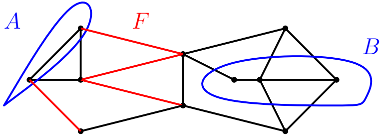

Definition

Given a graph G and a non-negative integer ℓ, we say that G is *ℓ-edge-connected* if $|V(G)| \geq 2$ and for all $F \subseteq E(G)$ s.t. $|F| \leq \ell - 1$, we have that $G \setminus F$ is connected.

Part II: Menger's theorems

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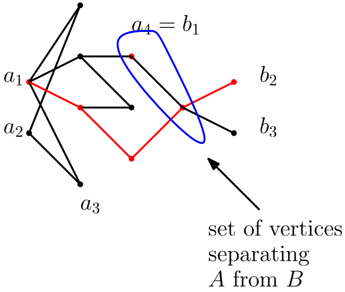
Menger's theorem (vertex version)

Let G be a graph, and let $A, B \subseteq V(G)$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of pairwise disjoint A-B paths in G.



$A = \{a_1, a_2, a_3, a_4\}$

$B = \{b_1, b_2, b_3\}$

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Proof (outline).

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Proof (outline). Assume inductively that the theorem is true for graphs on fewer than $|E(G)|$ edges. Let k be the minimum number of vertices separating A from B in G.

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(i) there can be no more than k pairwise disjoint paths from A to B in G;

(ii) there are at least k pairwise disjoint paths from A to B.

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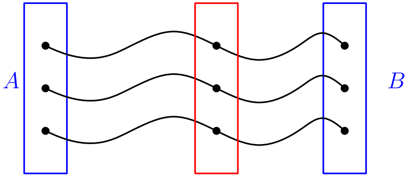
(i) is "obvious."



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Proof (outline, continued). Let's prove (ii):

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If $E(G) = \emptyset$, then $|A \cap B| = k$, and there are k pairwise disjoint A-B paths in G.

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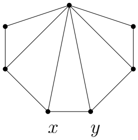
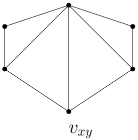
We apply the induction hypothesis to $G_{xy} := G/xy$.



G



G/xy

Menger's theorem (vertex version)

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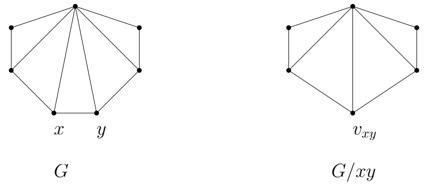
Proof (outline, continued).

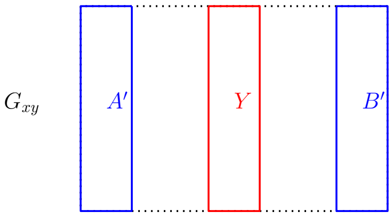


If x or y belongs to A, then let $A' = (A \setminus \{x, y\}) \cup \{v_{xy}\}$, and otherwise, let $A' = A$. Similarly, if x or y belongs to B, then let $B' = (B \setminus \{x, y\}) \cup \{v_{xy}\}$, and otherwise, let $B' = B$.

Menger's theorem (vertex version)

Let G be a graph, and let $A, B \subseteq V(G)$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of pairwise disjoint A-B paths in G.

Proof (outline, continued). Let $Y \subseteq V(G_{xy})$ be a minimum-sized set of vertices separating A' from B' in G_{xy}. By the induction hypothesis, there are $|Y|$ many pairwise disjoint paths in G_{xy} from A' to B', and it readily follows that there are at least $|Y|$ many pairwise disjoint paths in G from A to B. So, if $|Y| \geq k$, then we are done.



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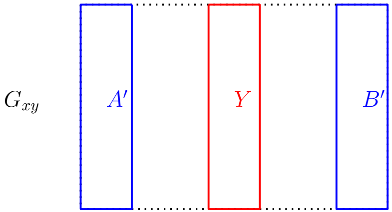
Let G be a graph, and let $A, B \subseteq V(G)$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of pairwise disjoint A-B paths in G.

Proof (outline, continued). From now on, we assume that $|Y| \leq k-1$. Then $v_{xy} \in Y$, for otherwise, Y would separate A from B in G, contrary to the fact that $|Y| \leq k-1$. Now $X := (Y \setminus \{v_{xy}\}) \cup \{x, y\}$ separates A from B in G, and we have that $|X| = |Y| + 1$. Note that this implies that $|X| = k$. Set $X = \{x_1, \dots, x_k\}$.



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Proof (outline, continued). We now consider the graph $G \setminus xy$.

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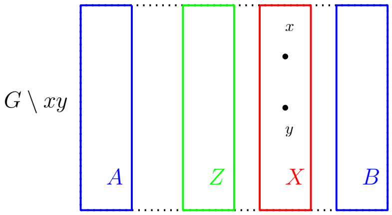
Proof (outline, continued). We now consider the graph $G \setminus xy$. Since $x, y \in X$, we know that any set of vertices separating A from X in $G \setminus xy$ also separates A from B in G; consequently, any such set has at least k vertices.



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Proof (outline, continued). So, by the induction hypothesis, there are k pairwise disjoint paths from A to X in G, call them $P_1, \ldots, P_k$.

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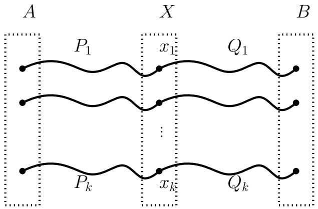
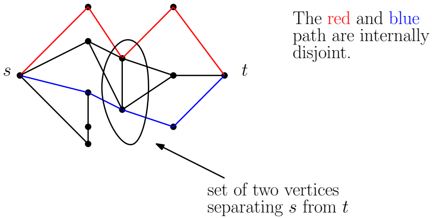
Proof (outline, continued). So, by the induction hypothesis, there are k pairwise disjoint paths from A to X in G, call them $P_1, \ldots, P_k$. Similarly, there are k pairwise disjoint paths from B to X in G, call them $Q_1, \ldots, Q_k$.



Corollary 1.1

Let G be a graph, and let $s, t \in V(G)$ be distinct, non-adjacent vertices of G. Then the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G.



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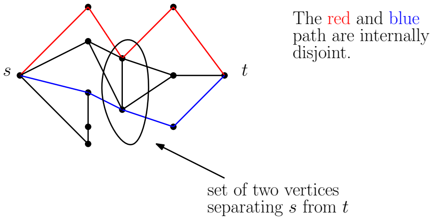
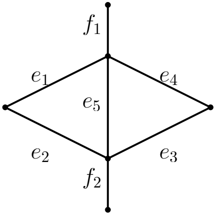


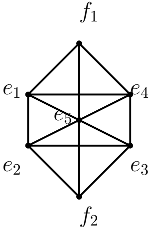
Proof (outline). Apply Menger's theorem (vertex version) to the graph $G \setminus \{s, t\}$ and sets $S = N_G(s)$ and $T = N_G(t)$.

Definition

The *line graph* of a graph G, denoted by $L(G)$, is the graph whose vertex set is $E(G)$, and in which $e, f \in L(V(G)) = E(G)$ are adjacent iff e and f share an endpoint in G.



G

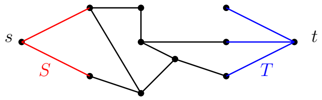


$L(G)$

Menger's theorem (edge version)

Let G be a graph, and let $s, t \in V(G)$ be distinct vertices of G. Then the minimum number of edges separating s from t in G is equal to the maximum number of pairwise edge-disjoint s-t paths in G.

Proof (outline). Apply Menger's theorem (vertex version) to the graph $L(G)$ and the sets $S = \{e \in E(G) \mid e \text{ is incident with } s\}$ and $T = \{e \in E(G) \mid e \text{ is incident with } t\}$.



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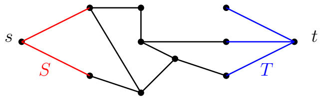

- The maximum number of pairwise edge-disjoint s-t paths in G is equal to the maximum number of pairwise disjoint S-T paths in $L(G)$.

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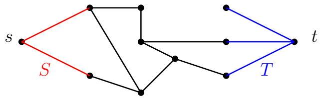

- The maximum number of pairwise edge-disjoint s-t paths in G is equal to the maximum number of pairwise disjoint S-T paths in $L(G)$.
- A set $X \subseteq E(G)$ separates s from t in G iff X separates S from T in $L(G)$.

The global version of Menger's theorem

Let G be a graph on ≥ 2 vertices, and let $k, \ell \geq 0$ be integers.

(a) G is k-connected iff for all distinct $s, t \in V(G)$, there are k pairwise internally disjoint s-t paths in G.

(b) G is ℓ-edge-connected iff for all distinct $s, t \in V(G)$, there are ℓ pairwise edge-disjoint s-t paths in G.

Proof.

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Proof. (b) is easier, and so we prove it first.

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Proof (continued).

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Proof (continued). Suppose that G is ℓ-edge-connected. Fix distinct vertices $s, t \in V(G)$. Since G is ℓ-edge-connected, s cannot be separated from t with fewer than ℓ edges of G, and so by Menger's theorem (edge version), there are at least ℓ pairwise edge-disjoint paths between s and t in G.

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Proof (continued). Suppose now that G is not ℓ-edge-connected.

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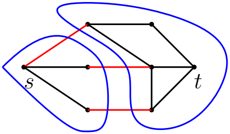
Proof (continued). Suppose now that G is not ℓ-edge-connected. Then $\exists F \subseteq E(G)$ s.t. $|F| \leq \ell - 1$ and $G \setminus F$ is disconnected. Let s, t be vertices from distinct components of $G \setminus F$.



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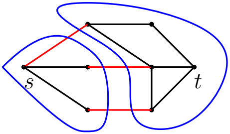

Now F separates s from t, and so s can be separated from t by at most $|F| \leq \ell - 1$ edges of G. So, by Menger's theorem (edge version), there are at most $\ell - 1$ pairwise edge-disjoint paths between s and t in G.

Corollary 1.1

Let G be a graph, and let $s, t \in V(G)$ be distinct, non-adjacent vertices of G. Then the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G.

The global version of Menger's theorem

Let G be a graph on ≥ 2 vertices, and let $k, \ell \geq 0$ be integers.

(a) G is k-connected iff for all distinct $s, t \in V(G)$, there are k pairwise internally disjoint s-t paths in G.

Proof (continued). It remains to prove (a).

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(a) G is k-connected iff for all distinct $s, t \in V(G)$, there are k pairwise internally disjoint s-t paths in G.

Proof (continued). It remains to prove (a). Suppose that G is k-connected, and let s and t be distinct vertices of G.

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Proof (continued). It remains to prove (a). Suppose that G is k-connected, and let s and t be distinct vertices of G. Suppose first that s and t are non-adjacent. Since G is k-connected, s and t cannot be separated by fewer than k vertices of $V(G) \setminus \{s, t\}$; so, by Corollary 1.1, there are k internally disjoint paths between s and t.

Corollary 1.1

Let G be a graph, and let $s, t \in V(G)$ be distinct, non-adjacent vertices of G. Then the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G.

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Proof (continued). Suppose now that s and t are adjacent.

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It remains to show that for all sets $X \subseteq V(G)$ s.t. $|X| \leq k-1$, we have that $G \setminus X$ is connected.

Corollary 1.1

Let G be a graph, and let $s, t \in V(G)$ be distinct, non-adjacent vertices of G. Then the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G.

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Part III: The structure of 2-connected graphs and the Ear lemma

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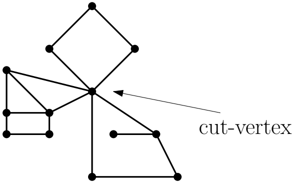
Definition

A *cut-vertex* of a graph G is any vertex $v \in V(G)$ s.t. $G \setminus v$ has more components than G.



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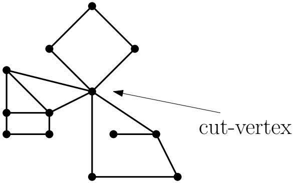

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For a non-negative integer k, a graph G is *k-connected* if $|V(G)| \geq k+1$ and for all $S \subseteq V(G)$ s.t. $|S| \leq k-1$, we have that $G \setminus S$ is connected.

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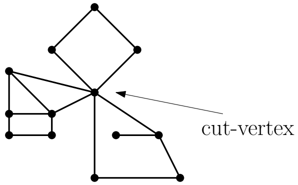

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- So, a graph is 2-connected if it has at least three vertices, is connected, and has no cut-vertices.

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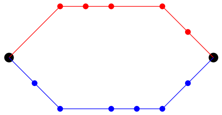
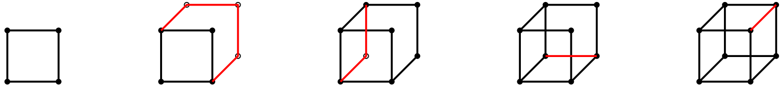
Proof. By Menger's theorem (global version), a graph on at least two vertices is 2-connected iff for any pair of distinct vertices, there are two internally disjoint paths between them. But obviously, two distinct vertices lie on a common cycle iff there are two internally-disjoint paths between them. The result now follows.



Definition

A *path addition* (sometimes called *open ear addition*) to a graph H is the addition to H of a path between two distinct vertices of H in such a way that no internal vertex and no edge of the path belongs to H.



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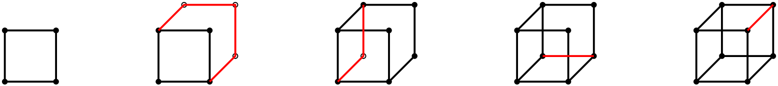

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Proof of the "$\Longleftarrow$" part. Clearly, cycles are 2-connected (indeed, every cycle has at least three vertices, is connected, and has no cut-vertices). Further, if a graph G can be obtained from a 2-connected graph H by adding a path, then G has at least three vertices (because H does), and it is easy to see that G is connected and has no cut-vertices; so, G is 2-connected.

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Proof of the "$\Longrightarrow$" part (outline). Fix a 2-connected graph G. By Lemma 1.1, G contains a cycle. Now, let H be a maximal subgraph of G that either is a cycle or can be obtained from a cycle by repeated path addition. We must show that $H = G$.

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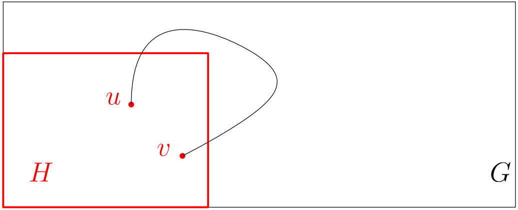
H is an induced subgraph of G, because otherwise, we can add another path to H.



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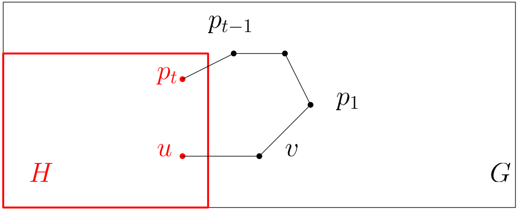
Proof of the "$\Longrightarrow$" part (outline, continued). Also, $V(H) = V(G)$, for otherwise, we could add another path to H.



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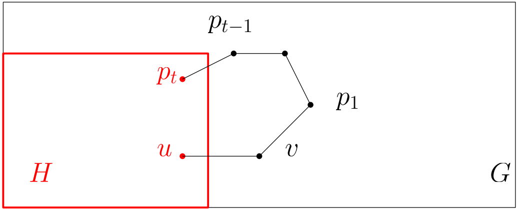

We now have that $V(H) = V(G)$, and that H is an induced subgraph of G. So, $H = G$.