NDMI011: Combinatorics and Graph Theory 1

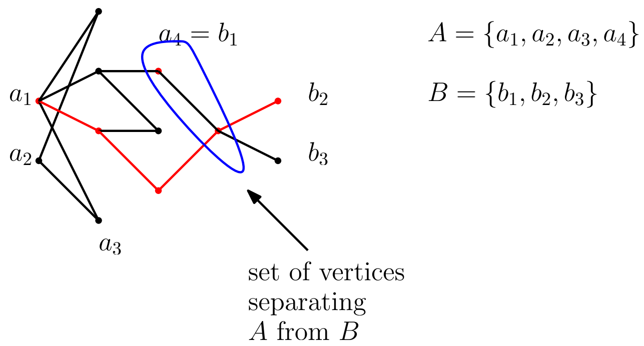
Lecture #8
Menger's theorems and the Ear lemma

Irena Penev

In what follows, all graphs are finite, simple (i.e. have no loops and no parallel edges), and non-null.

1 Menger's theorems

Menger's theorem (vertex version). *Let G be a graph, and let $A, B \subseteq V(G)$.*[1] *Then the minimum number of vertices separating A from B in G is equal to the maximum number of pairwise disjoint A-B paths in G.*[2]



Proof. We assume inductively that the theorem holds for graphs that have fewer than $|E(G)|$ edges. More precisely, we assume that for all graphs G' such that $|E(G')| < |E(G)|$, and all sets $A', B' \subseteq V(G')$, the minimum number of vertices separating A' from B' in G' is equal to the maximum number of pairwise disjoint A'-B' paths in G'. We must prove that this holds for G as well. From now on, we let k be the minimum number of vertices separating A from B in G.

[1] A and B need not be disjoint.

[2] "Pairwise disjoint" here means that no two paths have a vertex in common (and consequently, no two paths have an edge in common).

First, we claim that there can be no more than k pairwise disjoint paths from A to B in G. Indeed, let $X \subseteq V(G)$ be a k-vertex set separating A from B in G, and let $\mathcal{P}$ be any collection of pairwise disjoint paths from A to B. By definition, every path in $\mathcal{P}$ contains at least one vertex of X, and since paths in $\mathcal{P}$ are pairwise disjoint, no two paths in $\mathcal{P}$ contain the same vertex of X. So, $|\mathcal{P}| \leq |X| = k$, as we had claimed.

It remains to show that there are at least k pairwise disjoint paths from A to B. Clearly, for any set $X \subseteq V(G)$ separating A from B in G, we have that $A \cap B \subseteq X$; consequently, $|A \cap B| \leq k$. Now, if $E(G) = \emptyset$, then $A \cap B$ separates A from B in G, and so $|A \cap B| = k$; in this case, the vertices of $A \cap B$ form k pairwise disjoint one-vertex paths from A to B, and we are done. From now on, we assume that G has at least one edge, say xy. Let $G_{xy} := G/xy$, i.e. let G_{xy} be the graph obtained from G by contracting the edge xy, and let v_{xy} be the vertex obtained by contracting xy.[3]

G $\qquad$ G/xy

Now, if x or y belongs to A, then let $A' = (A \setminus \{x, y\}) \cup \{v_{xy}\}$, and otherwise, let $A' = A$. Similarly, if x or y belongs to B, then let $B' = (B \setminus \{x, y\}) \cup \{v_{xy}\}$, and otherwise, let $B' = B$.

Let $Y \subseteq V(G_{xy})$ be a minimum-sized set of vertices separating A' from B' in G_{xy}.[4] By the induction hypothesis, there are $|Y|$ many pairwise disjoint paths in G_{xy} from A' to B', and it readily follows[5] that there are at least $|Y|$ many pairwise disjoint paths in G from A to B. So, if $|Y| \geq k$,[6] then we are done. From now on, we assume that $|Y| \leq k - 1$. Then $v_{xy} \in Y$, for otherwise, Y would separate A from B in G,[7] contrary to the fact that $|Y| \leq k - 1$. Now $X := (Y \setminus \{v_{xy}\}) \cup \{x, y\}$ separates A from B in G,[8] and we have that $|X| = |Y| + 1$. Note that this implies that $|X| = k$.[9] Set

[3] Formally, v_{xy} is some ("new") vertex that does not belong to $V(G)$, and G_{xy} is the graph with vertex set $V(G_{xy}) = \Big(V(G) \setminus \{x, y\}\Big) \cup \{v_{xy}\}$ and edge set $E(G_{xy}) = \{e \in E(G) \mid e \text{ is incident neither with } x \text{ nor with } y \text{ in } G\} \cup \{vv_{xy} \mid v \in V(G), v \text{ is adjacent to } x \text{ or } y \text{ in } G\}$.

[4] This means that for all sets $Y' \subseteq V(G_{xy})$ separating A from B in G_{xy}, we have that $|Y| \leq |Y'|$.

[5] Details?

[6] In fact, it is not possible that $|Y| > k$ (details?), but we do not need this stronger fact.

[7] Proof?

[8] Proof?

[9] Indeed, since $|Y| \leq k - 1$, we have that $|X| \leq k$. On the other hand, since X separates A from B in G, we know that $|X| \geq k$. So, $|X| = k$.

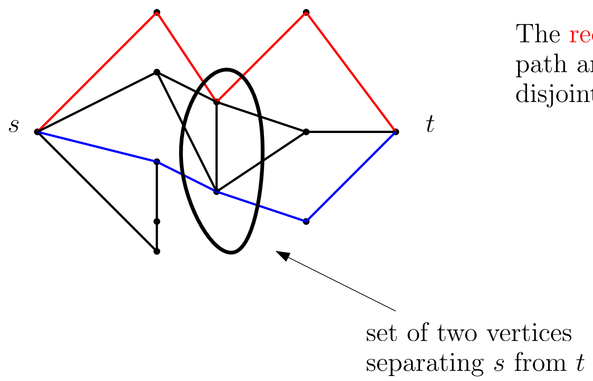
$X = \{x_1, \dots, x_k\}$.

We now consider the graph $G \setminus xy$, i.e. the graph obtained from G by deleting the edge xy.[10] Since $x, y \in X$, we know that any set of vertices separating A from X in $G \setminus xy$ also separates A from B in G;[11] consequently, any such set has at least k vertices, and so by the induction hypothesis, there are k pairwise disjoint paths from A to X in G, call them $P_1, \dots, P_k$. Similarly, there are k pairwise disjoint paths from B to X in G, call them $Q_1, \dots, Q_k$. We may assume that for all $i \in \{1, \dots, k\}$, x_i is an endpoint both of P_i and of Q_i. So, $P_1 - x_1 - Q_1, \dots, P_k - x_k - Q_k$ are pairwise disjoint walks from A to B. But in fact, each of these walks is a path, for otherwise, it would contain a path from A to B that contains no vertex of X.[12] So, there are at least k pairwise disjoint paths from A to B in G. $\square$

Given a graph G and distinct vertices $s, t \in V(G)$, two paths from s to t in G are *internally disjoint* if they have no vertices in common except the endpoints s and t.

The following corollary is also often referred to as the vertex version of Menger's theorem.

Corollary 1.1. *Let G be a graph, and let $s, t \in V(G)$ be distinct, non-adjacent vertices of G. Then the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G.*



The red and blue path are internally disjoint.

Proof. Let $S = N_G(s)$ and $T = N_G(t)$. Obviously, the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the minimum

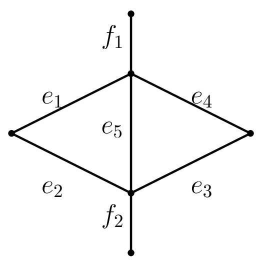
[10]So, $V(G \setminus xy) = V(G)$ and $E(G \setminus xy) = E(G) \setminus \{xy\}$.

[11]Let us check this. Let Z be any set of vertices separating A from X in $G \setminus xy$, and let $p_1, \dots, p_t$, with $p_1 \in A$ and $p_t \in B$, be a path from A to B in G. Then some vertex of $p_1, \dots, p_t$ belongs to X; let $i \in \{1, \dots, t\}$ be the smallest index such that $p_i \in X$. Then $p_1, \dots, p_i$ is a path from A to X in G. Furthermore, since $p_1, \dots, p_i$ contains exactly one vertex of X, and since $x, y \in X$, we see that the path $p_1, \dots, p_i$ does not use the edge xy; consequently, $p_1, \dots, p_i$ is a path from A to X in $G \setminus xy$, and we deduce that this path (and consequently, the path $p_1, \dots, p_t$ as well) contains a vertex of Z.

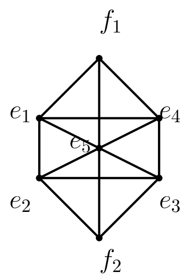
[12]Details?

number of vertices of $V(G) \setminus \{s,t\}$ separating S from T in $G \setminus \{s,t\}$.[13] Similarly, the maximum number of pairwise internally disjoint s-t paths in G is equal to the maximum number of pairwise disjoint S-T paths in G. By Menger's theorem (vertex version), the minimum number of vertices separating S from T in $G \setminus \{s,t\}$ is equal to the maximum number of pairwise disjoint S-T paths in $G \setminus \{s,t\}$. So, the minimum number of vertices of $V(G) \setminus \{s,t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G. This completes the argument. $\square$

Our next goal is to prove the edge version of Menger's theorem. The *line graph* of a graph G, denoted by $L(G)$, is the graph whose vertex set is $E(G)$, and in which $e, f \in L(V(G)) = E(G)$ are adjacent if and only if e and f share an endpoint in G.



G



$L(G)$

Proposition 1.2. *Let G be a graph, let $s, t \in V(G)$ be distinct vertices of G, let S be the set of all edges in G incident with s, and let T be the set of all edges in G incident with t. Let $X \subseteq E(G)$. Then X separates s from t in G if and only if X separates S from T in $L(G)$.*

Proof. Suppose that X separates s from t in G; we must show that X separates S from T in G. Suppose otherwise. Then there exists some path $e_1, \dots, e_r$ in $L(G)$ that does not contain any vertex (in $L(G)$) from X.[14] For each $i \in \{1, \dots, r-1\}$, let v_i be a common vertex of e_i and e_{i+1}.[15] Then $s, v_1, \dots, v_{r-1}, t$ is a walk in $L(G)$ from s to t that uses only edges $e_1, \dots, e_r$, and consequently, does not use any edge of X. It follows that there is a path from s to t in G that does not use any edges of X, contrary to the fact that X separates s from t in G. This proves that X indeed separates S from T in G.

Suppose now that X does not separate s from t in G; we must show that X does not separate S from T in $L(G)$. Since X does not separate s from t

[13] Indeed, for any set $X \subseteq V(G) \setminus \{s,t\}$, we have that X separates s from t in G if and only if X separates S from T in $G \setminus \{s,t\}$.

[14] Note that $e_1, \dots, e_r$ are vertices of $L(G)$, and consequently, edges of G.

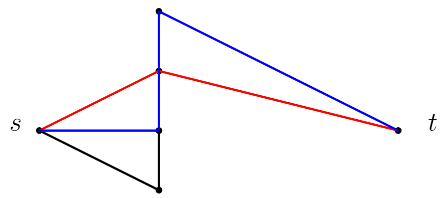
[15] Such a vertex exists because e_i and e_{i+1} are adjacent vertices of $L(G)$, and consequently, they are edges of G that share an endpoint.

in G, we know that there is a path $v_1, \dots, v_r$ in G, with $v_1 = s$ and $v_r = t$, that does not use any edge of X. But now $v_1v_2, v_2v_3, \dots, v_{r-1}v_r$ is a path from S to T in $L(G)$ that does not use any vertex (in $L(G)$) in X. So, X does not separate S from T in $L(G)$. $\square$

Proposition 1.3. *Let G be a graph, let $s, t \in V(G)$ be distinct vertices of G, let S be the set of all edges in G incident with s, and let T be the set of all edges in G incident with t. Let ℓ be a non-negative integer. Then the following are equivalent:*

(i) there are ℓ pairwise edge-disjoint s-t paths in G;

(ii) there are ℓ pairwise disjoint S-G paths in $L(G)$.

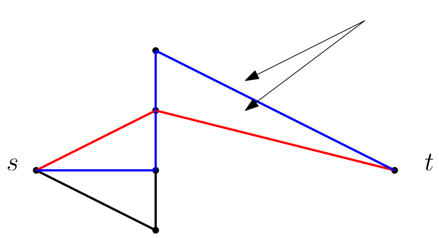


The red and blue path are edge-disjoint.

Proof. Suppose first that (i) holds, and fix ℓ pairwise edge-disjoint s-t paths in G, say $P_1, \dots, P_\ell$. For all $i \in \{1, \dots, \ell\}$, set $P_i = v_1^i, \dots, v_{r_i}^i$. Now, for all $i \in \{1, \dots, \ell\}$, set $P_i^L = v_1^i v_2^i, v_2^i v_3^i, \dots, v_{r_i-1}^i v_{r_i}^i$ (with $v_1^i = s$ and $v_{r_i}^i = t$). Clearly, $P_1^L, \dots, P_\ell^L$ are pairwise disjoint S-T paths in $L(G)$.

Suppose now that (ii) holds, and fix ℓ pairwise disjoint S-T paths in G, say $Q_1^L, \dots, Q_\ell^L$. For all $i \in \{1, \dots, \ell\}$, set $Q_i^L = e_1^i, \dots, e_{r_i}^i$. Now, for all $i \in \{1, \dots, \ell\}$ and $j \in \{1, \dots, r_i\}$, let v_j^i be a common vertex of the edges e_j^i and e_{j+1}^i in G, and set $Q_i = s, v_1^i, \dots, v_{r_i-1}^i, t$. Then $Q_1, \dots, Q_\ell$ are pairwise edge-disjoint s-t walks in G, and we deduce that there are ℓ pairwise edge-disjoint s-t paths in G. $\square$

Menger's theorem (edge version). *Let G be a graph, and let $s, t \in V(G)$ be distinct vertices of G. Then the minimum number of edges separating s from t in G is equal to the maximum number of pairwise edge-disjoint s-t paths in G.*



Proof. Let S be the set of all edges in G incident with s, and let T be the set of all edges in G incident with t. By Proposition 1.2, the minimum number of edges separating s from t in G is equal to the minimum number of vertices separating S from T in $L(G)$. By Proposition 1.3, the maximum number of pairwise edge-disjoint s-t paths in G is equal to the maximum number of pairwise disjoint S-T paths in G. By Menger's theorem (vertex version), the minimum number of vertices separating S from T in $L(G)$ is equal to the maximum number of pairwise disjoint S-T paths in G. We now deduce that the minimum number of edges separating s from t in G is equal to the maximum number of pairwise edge-disjoint s-t paths in G. This completes the argument. □

The global version of Menger's theorem. *Let G be a graph on at least two vertices, and let $k, \ell \geq 0$ be integers.*

(a) G is k-connected if and only if for all distinct $s, t \in V(G)$, there are k pairwise internally disjoint s-t paths in G.

(b) G is ℓ-edge-connected if and only if for all distinct $s, t \in V(G)$, there are ℓ pairwise edge-disjoint s-t paths in G.

Proof. We first prove (a). Suppose that G is k-connected, and let s and t be distinct vertices of G.

Suppose first that s and t are non-adjacent. Since G is k-connected, s and t cannot be separated by fewer than k vertices of $V(G) \setminus \{s, t\}$; so, by Corollary 1.1, there are k internally disjoint paths between s and t.

Suppose now that s and t are adjacent. Set $G' = G \setminus st$.[16] By Proposition 3.1 from Lecture Notes 7, G' is $(k-1)$-connected. Now s and t are distinct and non-adjacent in G', and they cannot be separated (in G') by fewer than $k-1$ vertices of $V(G') \setminus \{s, t\}$; so, Corollary 1.1 guarantees that there are $k-1$ internally disjoint paths between s and t in G'. These $k-1$ paths, plus the one-edge path s, t form k internally disjoint paths in G.

Suppose now that there are k internally disjoint paths between any two distinct vertices of G; we must show that G is k-connected.

Let us first show that $|V(G)| \geq k+1$. By hypothesis, G has at least two vertices; fix any distinct vertices $s, t \in V(G)$. Then there are k internally disjoint paths between them, and all but possibly one of those paths have an internal vertex;[17] so these k paths together have at least $k-1$ internal vertices, and it follows that $|V(G)| \geq (k-1)+2 = k+1$,[18] which is what we needed.

It remains to show that for all sets $X \subseteq V(G)$ such that $|X| \leq k-1$, we have that $G \setminus X$ is connected. Suppose otherwise, and fix some $X \subseteq V(G)$

[16] So, G' is the graph obtained from G by deleting the edge st.

[17] If s and t are adjacent, then s, t is a path between s and t with no internal vertices. However, any other path between s and t has at least one internal vertex.

[18] We are counting the $k-1$ internal vertices of our paths, plus the endpoints s and t

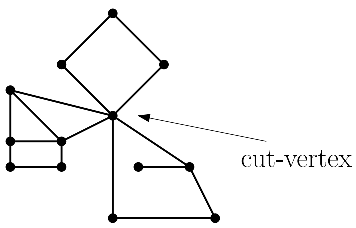
such that $|X| \leq k-1$ and $G \setminus X$ is disconnected. Then $G \setminus X$ has at least two components, and we choose vertices s and t from distinct components of $G \setminus X$. Now $X \subseteq V(G) \setminus \{s,t\}$ separates s from t, and so by Corollary 1.1, there can be at most $|X| \leq k-1$ internally disjoint paths between s and t in G. But this contradicts the fact that there are k internally disjoint paths between any two distinct vertices of G.

We now prove (b). Suppose that G is ℓ-edge-connected. Fix distinct vertices $s, t \in V(G)$. Since G is ℓ-edge-connected, s cannot be separated from t with fewer than ℓ edges of G, and so by Menger's theorem (edge version), there are at least ℓ pairwise edge-disjoint paths between s and t in G.

Suppose now that G is not ℓ-edge connected. Then there exists a set $F \subseteq E(G)$ such that $|F| \leq \ell - 1$ and $G \setminus F$ is disconnected. Since $G \setminus F$ is disconnected, it has at least two components; let s and t be vertices from distinct components of $G \setminus F$. Now F separates s from t, and in particular, s can be separated from t by at most $|F| \leq \ell - 1$ edges of G. So, by Menger's theorem (edge version), there are at most $\ell - 1$ pairwise edge-disjoint paths between s and t in G. □

2 2-connected graphs and ear decomposition

A *cut-vertex* of a graph G is any vertex $v \in V(G)$ such that $G \setminus v$ has more components than G.

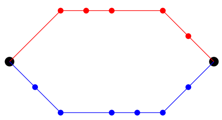


Recall that, for a non-negative integer k, a graph G is *k-connected* if $|V(G)| \geq k+1$ and for all $S \subseteq V(G)$ such that $|S| \leq k-1$, we have that $G \setminus S$ is connected. So, a graph is 2-connected if it has at least three vertices, is connected, and has no cut-vertices.

Lemma 2.1. *Let G be a graph on at least two vertices. Then G is 2-connected if and only if any two distinct vertices lie on a common cycle.*[19]

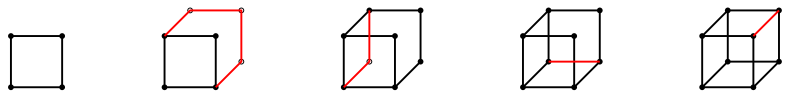
Proof. By Menger's theorem (global version), a graph on at least two vertices is 2-connected if and only if for any pair of distinct vertices, there are two internally disjoint paths between them. But obviously, two distinct vertices lie on a common cycle if and only if there are two internally-disjoint paths between them. The result now follows.

[19]Note that if G has at least two vertices, and any two distinct vertices lie on a common cycle, then in particular, G contains a cycle, and therefore, G has at least three vertices.



$\square$

In this section, we give a full structural description of 2-connected graphs. A *path addition* (sometimes called *open ear addition*) to a graph H is the addition to H of a path between two distinct vertices of H in such a way that no internal vertex and no edge of the path belongs to H. In the picture below, we show how the cube graph can be constructed by starting with a cycle of length four and then repeatedly adding open ears (the path/open ear added at each step is in red).



The Ear lemma. *A graph is 2-connected if and only if it is a cycle or can be obtained from a cycle by repeated path addition.*

Proof. We first prove the "if" (i.e. "$\Longleftarrow$") part of the lemma. Clearly, cycles are 2-connected (indeed, every cycle has at least three vertices, is connected, and has no cut-vertices).[20] Further, if a graph G can be obtained from a 2-connected graph H by adding a path, then G has at least three vertices (because H does), and it is easy to see that G is connected and has no cut-vertices;[21] so, G is 2-connected. It now follows by an easy induction (e.g. on the number of paths added) that any graph obtained from a cycle by repeated path addition is 2-connected. This proves the "if" part of the lemma.

It remains to prove the "only if" (i.e. "$\Longrightarrow$") part of the lemma. Fix a 2-connected graph G. By Lemma 2.1, G contains a cycle.[22] Now, let H be a maximal subgraph of G that either is a cycle or can be obtained from a cycle by repeated path addition.[23] We must show that $H = G$.

First, we claim that H is an induced subgraph of G.[24] If not, then there exist distinct vertices $u, v \in V(H)$ that are adjacent in G, but not in H; but then the graph obtained from H by adding the one-edge path u, v contradicts the maximality of H. So, H is indeed an induced subgraph of G.

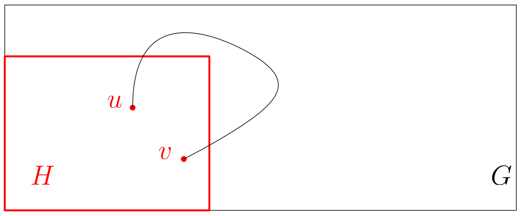
[20] Alternatively, this follows from Lemma 2.1.

[21] Check this!

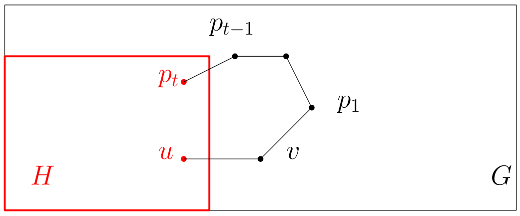
[22] Indeed, G has at least three vertices (because it is 2-connected), and by Lemma 2.1, any two of them lie on a common cycle. So, G contains a cycle.

[23] This means that no subgraph H^* of G that either is a cycle or can be obtained from a cycle by repeated path addition contains H as a proper subgraph.

[24] A graph H is an *induced subgraph* of a graph G if $V(H) \subseteq V(G)$, and for all distinct $u, v \in V(H)$, we have that $uv \in E(H)$ if and only if $uv \in E(G)$.



It remains to show that $V(H) = V(G)$. Suppose otherwise. Then since G is connected, there is at least one edge between $V(H)$ and $V(G) \setminus V(H)$; fix adjacent vertices $u \in V(H)$ and $v \in V(G) \setminus V(H)$. Since both G is 2-connected, we know that $G \setminus u$ is connected; consequently, there is a path in $G \setminus u$ from v to some vertex in $V(H) \setminus \{u\}$; let $P = v, p_1, \dots, p_t$ ($t \geq 1$) be a path in $G \setminus u$ with $p_t \in V(H) \setminus \{u\}$; we may assume that $p_1, \dots, p_{t-1} \in V(G) \setminus V(H)$.[25] But now the graph obtained from H by adding the path $u, v, p_1, \dots, p_t$ contradicts the maximality of H.



This proves that $V(H) = V(G)$. Since we already know that H is an induced subgraph of G, it follows that $H = G$. This proves the "only if" part of the lemma. $\square$

[25]Otherwise, we fix a minimal index $i \in \{1, \dots, t-1\}$ such that $p_i \in V(H)$, and we consider the path $v, p_1, \dots, p_i$ instead of $v, p_1, \dots, p_t$.