NDMI011: Combinatorics and Graph Theory 1

Lecture #7

Applications of networks. Graph connectivity

Irena Penev

November 10, 2021

- In what follows, all graphs are finite, simple (i.e. have no loops and no parallel edges), and non-null.

This lecture consists of three parts:

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1. Matchings.

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2. Latin rectangles.

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1. Matchings.
2. Latin rectangles.
3. An introduction to connectivity.

Part I: Matchings

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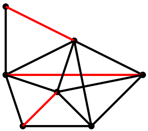
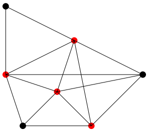
Definition

A *matching* in a graph G is a set of edges $M \subseteq E(G)$ such that every vertex of G is incident with at most one edge in M.



Definition

A *vertex cover* of a graph G is any set C of vertices of G such that every edge of G has at least one endpoint in C.



The Kőnig-Egerváry theorem

The maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover in that graph.

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(a) for every matching M and every vertex cover C of G, we have that $|M| \leq |C|$;

(b) there exist a matching M and a vertex cover C of G such that $|M| = |C|$.

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Proof of (a). Fix a matching M and a vertex cover C in G. Clearly, every edge of M has at least one endpoint in C. Since no two edges of M share an endpoint, we deduce that $|M| \leq |C|$. This proves (a).

Proof (continued).

(b) there exist a matching M and a vertex cover C of G such that $|M| = |C|$.

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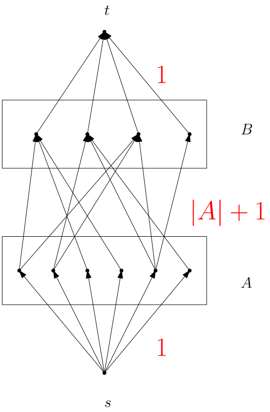
Proof of (b). We form a network (G', s, t, c) as follows:



Proof (continued).

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Proof of (b) (continued). Let f be a maximum flow in (G', s, t, c), and let R be a cut of minimum capacity.

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Proof of (b) (continued). Let f be a maximum flow in (G', s, t, c), and let R be a cut of minimum capacity. By Theorem 3.4 from Lecture Notes 6, we may assume that $f(e)$ is an integer for all $e \in E(G')$.

Proof (continued).

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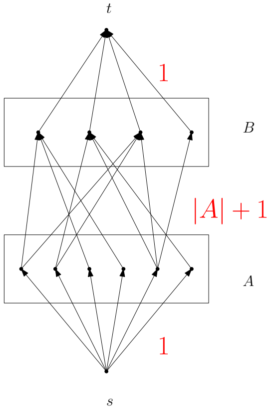
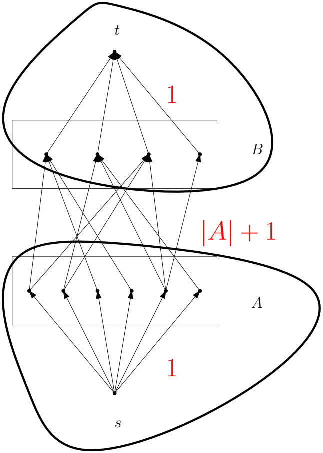
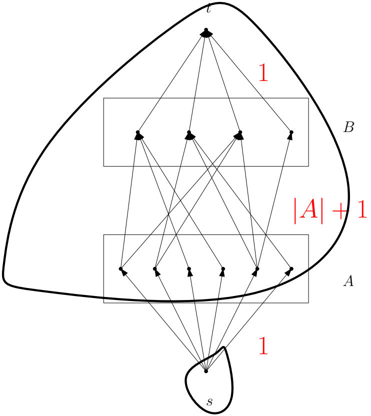
Proof of (b) (continued). Let f be a maximum flow in (G', s, t, c), and let R be a cut of minimum capacity. By Theorem 3.4 from Lecture Notes 6, we may assume that $f(e)$ is an integer for all $e \in E(G')$. By the Max-flow min-cut theorem, we know that $val(f) = c(R)$. It now suffices to produce a matching of size $val(f)$ and vertex cover of size $c(R)$.



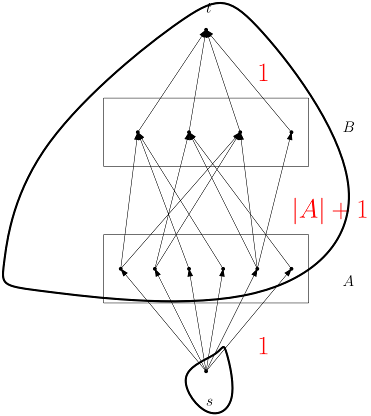
Proof (continued). Because of capacities, and because of inflows and outflows, we have that $f(e) \leq 1$ for all $e \in E(G')$. So, $f(e) \in \{0, 1\}$ for all $e \in E(G')$.



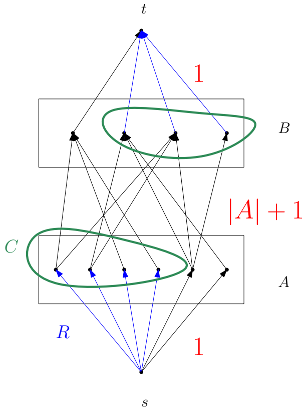
Proof (continued). Let
$M = \{ab \in E(G) \mid a \in A, b \in B, f(a,b) = 1\}$. Then M is a matching of size $val(f)$ (details: Lecture Notes).



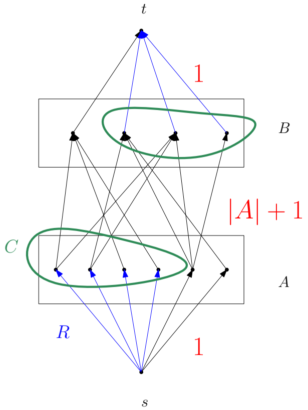
Proof (continued). Reminder: R is a cut of minimum capacity.



Proof (continued). Reminder: R is a cut of minimum capacity. R cannot contain any edges between A and B.



Proof (continued). Let C be the set of all vertices in $V(G) = A \cup B$ that are incident with at least one edge of R.



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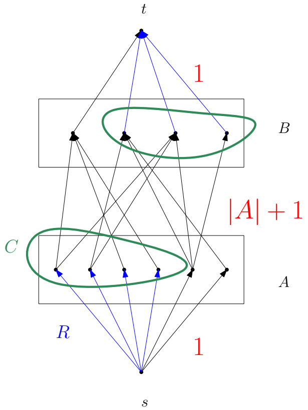
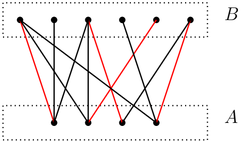


Proof (continued). Let C be the set of all vertices in $V(G) = A \cup B$ that are incident with at least one edge of R. Then $R = \{(s,a) \mid a \in A \cap C\} \cup \{(b,t) \mid b \in B \cap C\}$. It follows that $|C| = c(R)$, and C is a vertex cover of G (details: Lecture Notes).

Definition

Given a bipartite graph G with bipartition (A, B),

- an *A-saturating matching* in G is a matching M in G such that every vertex of A is incident with some edge in M;
- a *B-saturating matching* in G is a matching M in G such that every vertex of B is incident with some edge in M.



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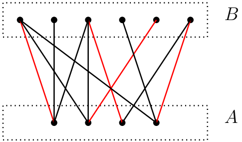


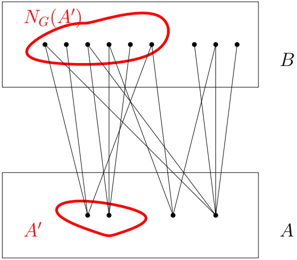
- For a graph G and a set $A \subseteq V(G)$, we denote by $N_G(A)$ the set of all vertices in $V(G) \setminus A$ that have a neighbor in A.

Hall's theorem (graph theoretic formulation)

Let G be a bipartite graph with bipartition (A, B). Then the following are equivalent:

(a) all sets $A' \subseteq A$ satisfy $|A'| \leq |N_G(A')|$;

(b) G has an A-saturating matching.



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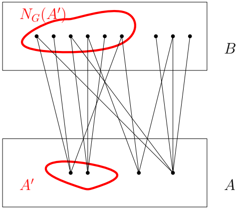

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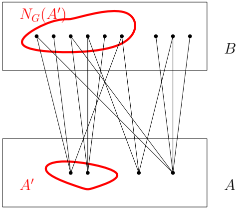

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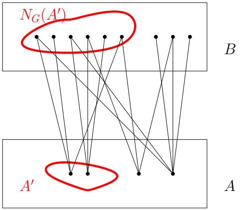
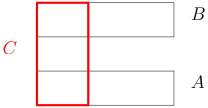


Proof (continued). "(b) $\Longrightarrow$ (a)." is "obvious." For "(a) $\Longrightarrow$ (b)," it suffices to show that any vertex cover of G is of size $\geq |A|$.

Proof (continued). Let C be a vertex cover of G.



Then there can be no edges between $A \setminus C$ and $B \setminus C$, and we deduce that $N_G(A \setminus C) \subseteq B \cap C$, and consequently, $|N_G(A \setminus C)| \leq |B \cap C|$. Now we have the following:

$$\begin{aligned} |A| &= |A \cap C| + |A \setminus C| \\ &\leq |A \cap C| + |N_G(A \setminus C)| \qquad \text{by (a)} \\ &\leq |A \cap C| + |B \cap C| \\ &= |C|. \end{aligned}$$

Corollary 1.1

Let G be a bipartite graph with bipartition (A, B). Assume that G has at least one edge and that for all $a \in A$ and $b \in B$, we have that $d_G(a) \geq d_G(b)$. Then G has an A-saturating matching.

Proof. Lecture Notes.

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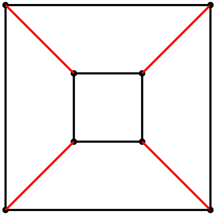
A *perfect matching* in a graph G is a matching M such that every vertex of G is incident with an edge in M.



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Hall's theorem (graph theoretic formulation)

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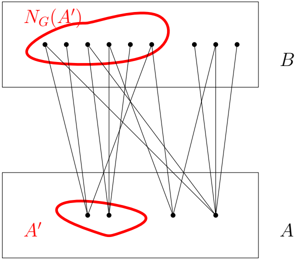

Definition

Suppose X and I are sets, and $\{A_i\}_{i \in I}$ is a family of (not necessarily distinct) subsets of X. A *transversal* (or a *system of distinct representatives*) for $(X, \{A_i\}_{i \in I})$ is an injective (i.e. one-to-one) function $f : I \to X$ such that for all $i \in I$, we have that $f(i) \in A_i$.

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Hall's theorem (combinatorial formulation)

Let X and I be finite sets, and let $\{A_i\}_{i\in I}$ be a family of (not necessarily distinct) subsets of X. Then the following are equivalent:

(a) all sets $J \subseteq I$ satisfy $|J| \leq |\bigcup_{j\in J} A_j|$;

(b) $(X, \{A_i\}_{i\in I})$ has a transversal.

Proof. Exercise.

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(a) for all sets $S \subsetneq V(G)$, we have that $\text{odd}(G \setminus S) \leq |S|$;

(b) G has a perfect matching.

Proof. Omitted.

Part II: Latin rectangles

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Definition

For positive integers r and n, with $r \leq n$, an $r \times n$ *Latin rectangle* is an $r \times n$ array (or matrix) whose entries are numbers $1, \ldots, n$, and in which each number $1, \ldots, n$ occurs at most once in each row and each column.

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Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $B = \{1, \dots, n\}$, and let G be the bipartite graph with bipartition (A, B) in which $\mathbf{a}_i \in A$ and $j \in B$ are adjacent if and only if j is not an entry of the column $\mathbf{a}_i$.

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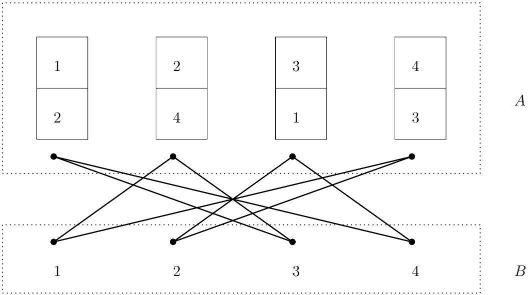

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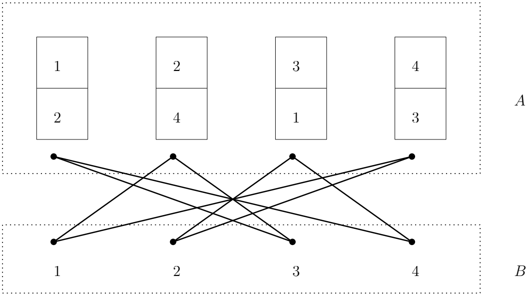

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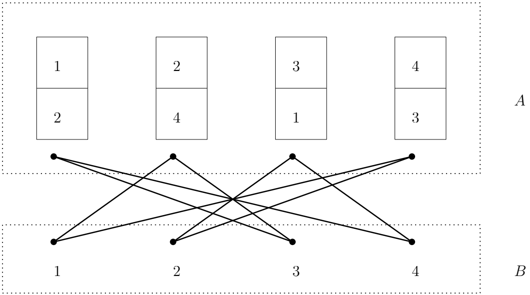

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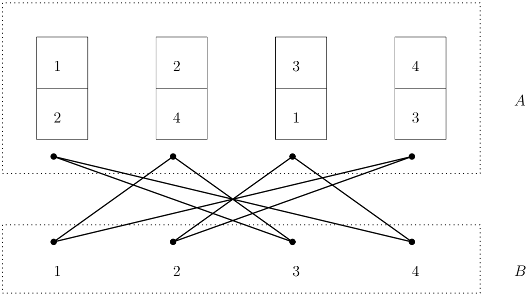

Then G is an $(n - r)$-regular bipartite graph that has at least one edge. So, by Corollary 1.2, G has a perfect matching. This perfect matching gives a "recipe" for adding one row to our $r \times n$ Latin rectangle in a way that produces an $(r + 1) \times n$ Latin rectangle.

Part III: An introduction to connectivity

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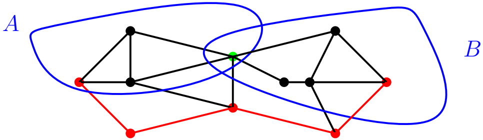
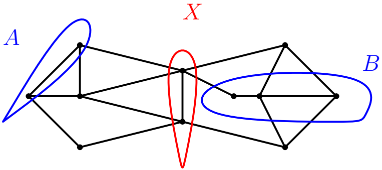
Definition

For a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, an *A-B path* in G, or a *path from A to B* in G, is either a one-vertex path whose sole vertex is in $A \cap B$, or a path on at least two vertices whose one endpoint is in A and whose other endpoint is in B.



Definition

Given a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, we say that a set $X \subseteq V(G)$ *separates* A from B in G if every path from A to B in G contains at least one vertex of X. Note that this implies that $A \cap B \subseteq X$.



Definition

Given a graph G and a non-negative integer k, we say that G is *k-vertex-connected*, or simply *k-connected*, if $|V(G)| \geq k+1$ and for all $X \subseteq V(G)$ such that $|X| \leq k-1$, we have that $G \setminus X$ is connected.

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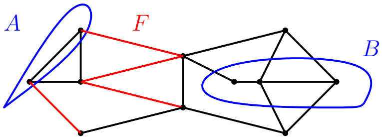
Given a graph G and disjoint sets $A, B \subseteq V(G)$, we say that a set $F \subseteq E(G)$ *separates* A from B in G if every path from A to B contains at least one edge of F.



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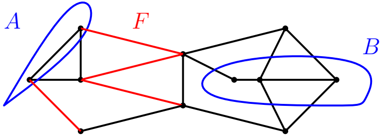

Definition

Given a graph G and a non-negative integer ℓ, we say that G is *ℓ-edge-connected* if $|V(G)| \geq 2$ and for all $F \subseteq E(G)$ such that $|F| \leq \ell - 1$, we have that $G \setminus F$ is connected.

Definition

The *edge-connectivity* of a graph G on at least two vertices, denoted by $\lambda(G)$, is the largest integer ℓ such that G is ℓ-edge-connected.

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- If there exists a set of at most ℓ edges whose deletion from G yields a disconnected graph, then $\lambda(G) \leq \ell$.

Proposition 3.1

Let G be a graph on at least two vertices. Then

(a) for all edges $e \in E(G)$, $\lambda(G) - 1 \leq \lambda(G \setminus e) \leq \lambda(G)$;
(b) for all sets $F \subseteq E(G)$, $\lambda(G \setminus F) \leq \lambda(G)$.

Proof. Lecture Notes.

Proposition 3.2

Let G be a graph on at least two vertices. Then

(a) for all edges $e \in E(G)$, $\kappa(G) - 1 \leq \kappa(G \setminus e) \leq \kappa(G)$;
(b) for all sets $F \subseteq E(G)$, $\kappa(G \setminus F) \leq \kappa(G)$.

Proof. Lecture Notes.

- However, unlike edge deletion, vertex deletion sometimes increases connectivity.

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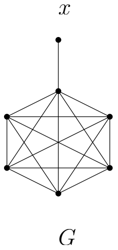
- For instance, for the graph G represented below, we have that $\kappa(G) = \lambda(G) = 1$, but $\kappa(G \setminus x) = \lambda(G \setminus x) = 5$.



Theorem 3.3

Let G be a graph on at least two vertices. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

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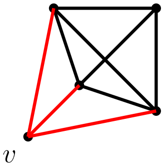
Proof. We first prove that $\lambda(G) \leq \delta(G)$. Fix a vertex $v \in V(G)$ such that $d_G(v) = \delta(G)$, and let F be the set of all edges of G that are incident with v. Clearly, $G \setminus F$ is disconnected, and it follows that $\lambda(G) \leq \delta(G)$.



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Proof (continued). It remains to show that $\kappa(G) \leq \lambda(G)$. Fix a set $F \subseteq E(G)$ such that $|F| = \lambda(G)$ and $G \setminus F$ is disconnected.

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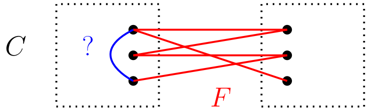
Proof (continued). It remains to show that $\kappa(G) \leq \lambda(G)$. Fix a set $F \subseteq E(G)$ such that $|F| = \lambda(G)$ and $G \setminus F$ is disconnected.

> ***Claim.*** *If C is the vertex set of a component of $G \setminus F$, then no edge of F has both its endpoints in C.*



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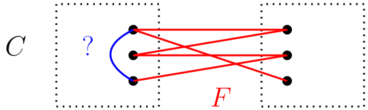

Proof of the Claim. Suppose some edge $e \in F$ be an edge that has both its endpoints in C. Then $G \setminus (F \setminus \{e\})$ is still disconnected, contrary to the fact that $|F \setminus \{e\}| = |F| - 1 = \lambda(G) - 1$. This proves the Claim.

Theorem 3.3

Let G be a graph on at least two vertices. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proof (continued). Reminder: $F \subseteq E(G)$, $|F| = \lambda(G)$, $G \setminus F$ is disconnected. WTS $\kappa(G) \leq \lambda(G)$.

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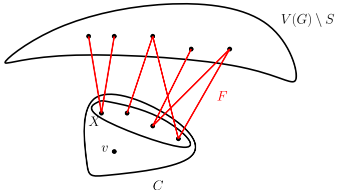
Proof (continued). Reminder: $F \subseteq E(G)$, $|F| = \lambda(G)$, $G \setminus F$ is disconnected. WTS $\kappa(G) \leq \lambda(G)$.

Suppose first that there exists a vertex $v \in V(G)$ that is not incident with any edge in F. Let C be the vertex set of the component of $G \setminus F$ that contains v. By the Claim, no edge in F has both endpoints in C. Now, let X be the set of all vertices in C that are incident with an edge in F. Then $|X| \leq |F| = \lambda(G)$ and $G \setminus X$ is disconnected. So, $\kappa(G) \leq \lambda(G)$.



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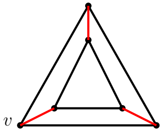
Suppose now that every vertex of G is incident with an edge of F.



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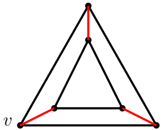

Let $v \in V(G)$, and let C be the component of $G \setminus F$ containing v. Then each vertex in $N_C[v]$ is incident with an edge of F, and (by the Claim) no two vertices of $N_C[v]$ are incident with the same edge of F.

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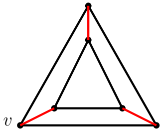

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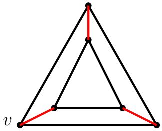

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Let G be a graph on at least two vertices. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proof (continued). Reminder: $F \subseteq E(G)$, $|F| = \lambda(G)$, $G \setminus F$ is disconnected, $\lambda(G) = \Delta(G)$. WTS $\kappa(G) \leq \lambda(G)$.

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Now, if G is a complete graph, then $|V(G)| = \Delta(G) + 1$, and we see that $\kappa(G) = \Delta(G) = \lambda(G)$. So assume that G is not complete, and fix some $x \in V(G)$ that has a non-neighbor in G. Then $G \setminus N_G(x)$ is disconnected, and we have that $|N_G(x)| = d_G(x) \leq \Delta(G) = \lambda(G)$. So, $\kappa(G) \leq \lambda(G)$.

Definition

A *vertex-cutset* of a graph G is any set $X \subsetneq V(G)$ such that $G \setminus X$ has more components than G. Similarly, an *edge-cutset* of G is any set $F \subseteq E(G)$ such that $G \setminus F$ has more components than G.

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- Similarly, no graph G has an edge-cutset of size strictly smaller than $\lambda(G)$.