NDMI011: Combinatorics and Graph Theory 1

Lecture #7
Applications of networks. Graph connectivity

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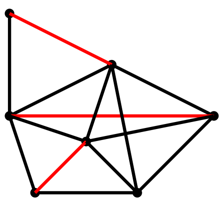
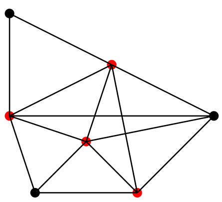
In what follows, all graphs are finite, simple (i.e. have no loops and no parallel edges), and non-null.

1 Matchings and transversals

A *matching* in a graph G is a set of edges $M \subseteq E(G)$ such that every vertex of G is incident with at most one edge in M. An example of a matching in a graph is given below (edges of the matching are in red).



A *vertex cover* of a graph G is any set C of vertices of G such that every edge of G has at least one endpoint in C. An example of a vertex cover in a graph is given below (vertices of the vertex cover are in red).



The Kőnig-Egerváry theorem. *The maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover in that graph.*

Proof. Let G be a bipartite graph with bipartition (A, B). Clearly, it suffices to prove the following two statements:

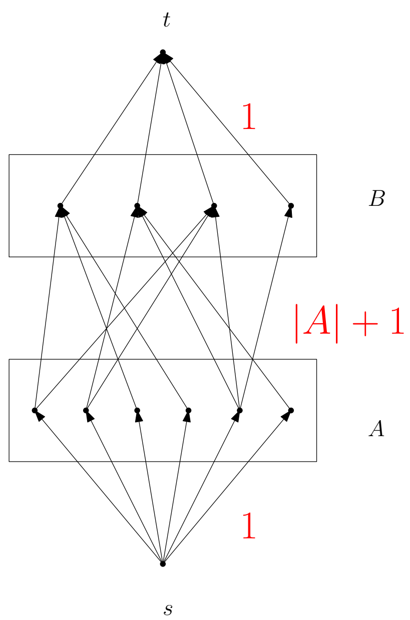
(a) for every matching M and every vertex cover C of G, we have that $|M| \le |C|$;[1]

(b) there exist a matching M and a vertex cover C of G such that $|M| = |C|$.

We begin by proving (a). Fix a matching M and a vertex cover C in G. Clearly, every edge of M has at least one endpoint in C. Since no two edges of M share an endpoint, we deduce that $|M| \le |C|$. This proves (a).

It remains to prove (b). Let s and t be two new vertices, i.e. $s \neq t$ and $s, t \notin V(G)$. We now form a network (G', s, t, c) as follows:

- $V(G') = V(G) \cup \{s, t\}$;
- $E(G') = \{(s, a) \mid a \in A\} \cup \{(a, b) \mid a \in A, b \in B, ab \in E(G)\} \cup \{(b, t) \mid b \in B\}$;
- $c(a, b) = |A| + 1$ for all $(a, b) \in E(G')$, with $a \in A$ and $b \in B$;
- $c(s, a) = 1$ for all $a \in A$;
- $c(b, t) = 1$ for all $b \in B$.



Let f be a maximum flow in (G', s, t, c), and let R be a cut of minimum capacity. By Theorem 3.4 from Lecture Notes 6, we may assume that $f(e)$ is an integer for all $e \in E(G')$. By the Max-flow min-cut theorem, we know that $val(f) = c(R)$. It now suffices to produce a matching of size $val(f)$ and vertex cover of size $c(R)$.

[1] In fact, (a) holds for all graphs, not just bipartite ones. However, there are (non-bipartite) graphs for which (b) fails.

First, we claim that $f(e) \in \{0,1\}$ for all $e \in E(G')$. Clearly, it suffices to show that $f(e) \leq 1$ for all $e \in E(G')$.[2] For all $a \in A$, we have that $f(s,a) \leq c(s,a) = 1$; and for all $b \in B$, we have that $f(b,t) \leq c(b,t) = 1$. Now, fix $a \in A$ and $b \in B$ such that $ab \in E(G)$. The inflow into a is at most 1,[3] and so the outflow is at most one. So, $f(a,b) \leq 1$. This proves that $f(e) \in \{0,1\}$ for all $e \in E(G')$, as we had claimed.

Now, let $M = \{ab \in E(G) \mid a \in A, b \in B, f(a,b) = 1\}$. Then[4]

$$\begin{aligned} |M| &= |\{(a,b) \in E(G') \mid a \in A, b \in B, f(a,b) = 1\}| \\ &= |\{e \in S_{G'}(A \cup \{s\}, B \cup \{t\}) \mid f(e) = 1\}| \\ &\overset{(*)}{=} f(A \cup \{s\}, B \cup \{t\}) \\ &\overset{(**)}{=} val(f), \end{aligned}$$

where (*) follows from the fact that $f(e) \in \{0,1\}$ for all $e \in E(G)$, and (**) follows from Lemma 2.3 from Lecture Notes 6. Let us check that M is a matching in G. Suppose otherwise. Then one of the following holds:

(i) there exist $a \in A$ and $b_1, b_2 \in B$ (with $b_1 \neq b_2$) such that $ab_1, ab_2 \in M$;

(ii) there exist $a_1, a_2 \in A$ (with $a_1 \neq a_2$) and $b \in B$ such that $a_1b, a_2b \in M$.

Suppose first that (i) holds. Then $f(a,b_1) = f(a,b_2) = 1$, and so the outflow from a is at least 2. On the other hand, the inflow into a is at most 1,[5] a contradiction. Suppose now that (ii) holds. then $f(a_1,b) = f(a_2,b) = 1$, and so the inflow into b is at least 2. On the other hand, the outflow from b is at most 1,[6] a contradiction. This proves that M is indeed a matching.

It remains to produce a vertex cover of size $c(R)$. Let C be the set of all vertices in $V(G) = A \cup B$ that are incident with at least one edge of R. Our goal is to show that C is a vertex cover of size at most $c(R)$. First, note that $\{(s,a) \mid a \in A\}$ is a cut in (G', s, t, c) of capacity $|A|$, and so $c(R) \leq |A|$. Since every edge from A to B has capacity $|A| + 1 > c(R)$, we deduce that R does not contain any edges from A to B; then $R = \{(s,a) \mid$

[2]This is because, for all $e \in E(G')$, $f(e)$ is a non-negative integer, and so if $f(e) \leq 1$, then $f(e) \in \{0,1\}$.

[3]This is because (s,a) is the only edge in G' with head a, and $f(s,a) \leq c(s,a) = 1$.

[4]$S_{G'}(A \cup \{s\}, B \cup \{t\})$ is the set of all edges from $A \cup \{s\}$ to $B \cup \{t\}$ in the oriented graph G'; note that all edges in $S_{G'}(A \cup \{s\}, B \cup \{t\})$ are in fact from A to B.

[5]This is because (s,a) is the only edge in G' with head a, and $f(s,a) \leq c(s,a) = 1$.

[6]This is because (b,t) is the only edge in G' with tail b, and $f(b,t) \leq c(b,t) = 1$.

$a \in A \cap C\} \cup \{(b,t) \mid b \in B \cap C\}$. It follows that

$$\begin{aligned} c(R) &= \Big(\sum_{a \in A \cap C} \underbrace{c(s,a)}_{=1}\Big) + \Big(\sum_{b \in B \cap C} \underbrace{c(b,t)}_{=1}\Big) \\ &= |A \cap C| + |B \cap C| \\ &= |C|. \end{aligned}$$

It remains to show that C is a vertex cover of G. Fix adjacent vertices $a \in A$ and $b \in B$; we must show that at least one of a, b belongs to C. Suppose otherwise. It then follows from the construction of C that R contains none of the edges (s,a), (a,b), and (b,t) of G', and consequently, s, a, b, t is a directed path from s to t in $G' \setminus R$, contrary to the fact that R is a cut in (G', s, t, c). This proves that C is indeed a vertex cover of G. This completes the proof of (b). $\square$

Given a bipartite graph G with bipartition (A, B),

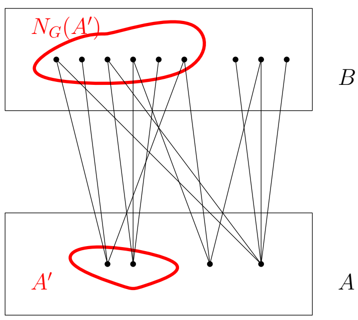
- an *A-saturating matching* in G is a matching M in G such that every vertex of A is incident with some edge in M;
- a *B-saturating matching* in G is a matching M in G such that every vertex of B is incident with some edge in M.

For a graph G and a set $A \subseteq V(G)$, we denote by $N_G(A)$ the set of all vertices in $V(G) \setminus A$ that have a neighbor in A. As a corollary of the Kőnig-Egerváry theorem, we obtain the following.

Hall's theorem (graph theoretic formulation). *Let G be a bipartite graph with bipartition (A, B). Then the following are equivalent:*

(a) all sets $A' \subseteq A$ satisfy $|A'| \leq |N_G(A')|$;

(b) G has an A-saturating matching.



Proof. Suppose first that (b) holds; we must prove that (a) holds. Fix an A-saturating matching M in G, and fix $A' \subseteq A$. Since M is an A-saturating matching, and since A' is a stable set,[7] we know that precisely $|A'|$ edges in M are incident with a vertex in A', and each of those edges has another endpoint in B. No two edges in M share an endpoint, and it follows that exactly $|A'|$ vertices in B are incident with an edge of M that has an endpoint in A'; let B' be the set of all such vertices of B. But clearly, $B' \subseteq N_G(A')$, and so $|N_G(A')| \geq |B'| = |A'|$. This proves (a).

Suppose, conversely, that (a) holds; we must prove that (b) holds. Since all edges of G are between A and B, it suffices to show that G has a matching of size at least $|A|$.[8] By the Kőnig-Egerváry theorem, it is enough to show that any vertex cover of G is of size at least $|A|$. Let C be a vertex cover of G. Then there can be no edges between $A \setminus C$ and $B \setminus C$, and we deduce that $N_G(A \setminus C) \subseteq B \cap C$, and consequently, $|N_G(A \setminus C)| \leq |B \cap C|$. Now we have the following:

$$\begin{aligned}
|A| &= |A \cap C| + |A \setminus C| \\
&\leq |A \cap C| + |N_G(A \setminus C)| \qquad \text{by (a)} \\
&\leq |A \cap C| + |B \cap C| \\
&= |C|.
\end{aligned}$$

This completes the proof of (b). □

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G that v is incident with.

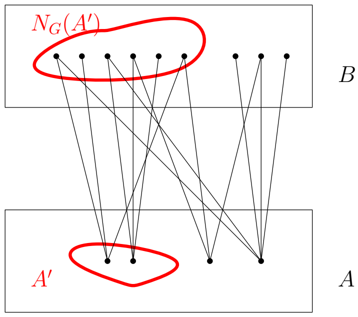
Corollary 1.1. *Let G be a bipartite graph with bipartition (A, B). Assume that G has at least one edge and that for all $a \in A$ and $b \in B$, we have that $d_G(a) \geq d_G(b)$. Then G has an A-saturating matching.*

Proof. We first check that $d_G(a) \geq 1$ for all $a \in A$. Suppose otherwise, and fix some $a_0 \in A$ such that $d(a_0) = 0$. Now, since G has at least one edge, and since every edge of G has one endpoint in A and the other one in B, we see that some vertex $b_0 \in B$ is incident with at least one edge, and so $d_G(b_0) \geq 1$. But now $d_G(a_0) < d_G(b_0)$, a contradiction. This proves that $d_G(a) \geq 1$ for all $a \in A$, as we had claimed.

Now, suppose that G does not have an A-saturating matching. Then by Hall's theorem, there exists some $A' \subseteq A$ such that $|A'| > |N_G(A')|$.

[7] A *stable set* (or *independent set*) is a set of pairwise non-adjacent vertices.

[8] Note that any matching in G of size at least $|A|$ is in fact of size precisely $|A|$.



Note that every edge in G has at least one endpoint in $(A \setminus A') \cup N_G(A')$,[9] and so

$$
\begin{aligned}
|E(G)| &\leq \sum_{v \in (A \setminus A') \cup N_G(A')} d_G(v) \\
&\leq \Big(\sum_{a \in A \setminus A'} d_G(a) \Big) + \Big(\sum_{b \in N_G(A')} d_G(b) \Big).
\end{aligned}
$$

Now, since $A' \subseteq A$ and $N_G(A') \subseteq B$, we know that for all $a \in A'$ and $b \in N_G(A')$, we have that $d_G(a) \geq d_G(b)$. Furthermore, by our choice of A', we have that $|A'| > |N_G(A')|$. Since $d_G(a) \geq 1$ for all $a \in A$, we now deduce that $\sum\limits_{a \in A'} d_G(a) > \sum\limits_{b \in N_G(A')} d_G(b)$, and it follows that

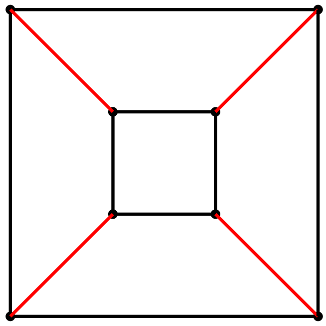
$$
\begin{aligned}
|E(G)| &\leq \Big(\sum_{a \in A \setminus A'} d_G(a) \Big) + \Big(\sum_{b \in N_G(A')} d_G(b) \Big). \\
&< \Big(\sum_{a \in A \setminus A'} d_G(a) \Big) + \Big(\sum_{a \in A'} d_G(a) \Big) \\
&= \sum_{a \in A} d_G(a).
\end{aligned}
$$

But this is impossible since, obviously, $|E(G)| = \sum\limits_{a \in A} d_G(a)$. □

For a non-negative integer k, a graph G is *k-regular* if it all its vertices are of degree k. G is *regular* if there exists some non-negative integer k such that G is k-regular.

A *perfect matching* in a graph G is a matching M such that every vertex of G is incident with an edge in M. An example of a perfect matching is shown below (edges of the perfect matching are in red).

[9]Indeed, if some edge of G had neither endpoint in $(A \setminus A') \cup N_G(A')$, then one of its endpoints would be in A' and the other one would be in $B \setminus N_G(A')$, a contradiction.



Obviously, not all graphs have perfect matchings. For instance, no graph with an odd number of vertices has a perfect matching. (There are also many graphs that have an even number of vertices, and yet do not have a perfect matching.)

Corollary 1.2. *Every regular bipartite graph that has at least one edge has a perfect matching.*

Proof. Let G be a k-regular ($k \geq 0$) bipartite graph with bipartition (A, B), and assume that G has at least one edge. By Corollary 1.1, G has an A-saturating matching. Now, since G has at least one edge, we see that $k \geq 1$. Further, since G is k-regular, we have that $|E(G)| = k|A|$ and $|E(G)| = k|B|$, and so $k|A| = k|B|$; since $k \neq 0$, it follows that $|A| = |B|$. Consequently, any A-saturating matching of G is a perfect matching. Since G has an A-saturating matching, it follows that G has a perfect matching. $\square$

For a graph G, let $\text{odd}(G)$ be the number of odd components (i.e. components with an odd number of vertices) of G. The following theorem gives a necessary and sufficient condition for a graph to have a perfect matching.

Tutte's theorem. *Let G be a graph. Then the following are equivalent:*

(a) for all sets $S \subsetneqq V(G)$, we have that $\text{odd}(G \setminus S) \leq |S|$;

(b) G has a perfect matching.

Proof. Omitted. $\square$

We complete this section by giving another formulation of Hall's theorem. We first need a definition. Suppose X and I are sets, and $\{A_i\}_{i \in I}$ is a family of (not necessarily distinct) subsets of X.[10] A *transversal* (or a *system of distinct representatives*) for $(X, \{A_i\}_{i \in I})$ is an injective (i.e. one-to-one) function $f : I \to X$ such that for all $i \in I$, we have that $f(i) \in A_i$.

Hall's theorem (combinatorial formulation). *Let X and I be finite sets, and let $\{A_i\}_{i \in I}$ be a family of (not necessarily distinct) subsets of X. Then the following are equivalent:*

[10] Technically, we have that $A : I \to \mathscr{P}(X)$; for $i \in I$, we write A_i instead of $A(i)$.

(a) all sets $J \subseteq I$ satisfy $|J| \leq |\bigcup_{j \in J} A_j|$;

(b) $(X, \{A_i\}_{i \in I})$ has a transversal.

Proof. Exercise. $\square$

2 Extending Latin rectangles

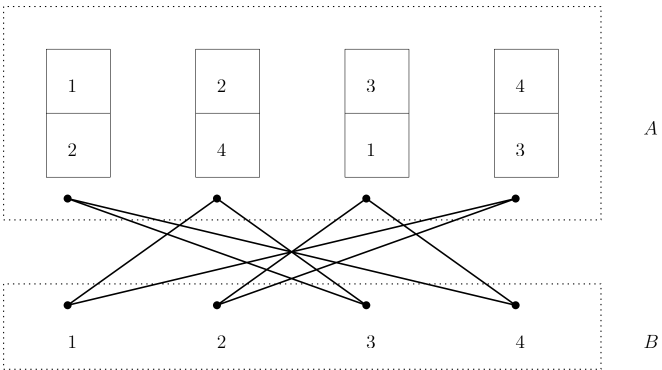
For positive integers r and n, with $r \leq n$, an $r \times n$ *Latin rectangle* is an $r \times n$ array (or matrix) whose entries are numbers $1, \dots, n$, and in which each number $1, \dots, n$ occurs at most once in each row and each column. One 2×4 Latin rectangle is represented below.

1	2	3	4
2	4	1	3

Theorem 2.1. *Let r and n be positive integers such that $r < n$. Then every $r \times n$ Latin rectangle can be extended to an $n \times n$ Latin square.*[11]

Proof. Let $L = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$ be an $r \times n$ Latin rectangle.[12] Obviously, it suffices to show that we can extend L to an $(r+1) \times n$ Latin rectangle by adding a row of length n to the bottom of L, for then the result will follow immediately by an easy induction.

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $B = \{1, \dots, n\}$, and let G be the bipartite graph with bipartition (A, B) in which $\mathbf{a}_i \in A$ and $j \in B$ are adjacent if and only if j is not an entry of the column $\mathbf{a}_i$. For instance, for the Latin rectangle from the beginning of the section, we would get the following bipartite graph:



[11]This means that, for any $r \times n$ Latin rectangle, it is possible to add $n - r$ rows of length n to the bottom of the Latin rectangle that we started with and thus obtain an $n \times n$ Latin square.

[12]This means that $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the columns of our Latin rectangle, in that order.

Each column of L has r entries, and consequently, there are $n-r$ values in B that do not appear in it. So, for all $\mathbf{a}_i \in A$, we have that $d_G(\mathbf{a}_i) = n-r$. Now, fix $j \in B$. We know that j appears exactly once in each row of L, and L has r rows. Consequently, j appears exactly r times in L, and since it cannot appear more than once in any column, we see that it appears in precisely r columns of L. Thus, j fails to appear in precisely $n-r$ columns of L, and consequently, $d_G(j) = n-r$. We have now shown that is $(n-r)$-regular. So, G is a regular bipartite graph, and (since $r < n$) it has at least one edge. Corollary 1.2 now implies that G has a perfect matching, call it M. Now, for each $i \in \{1, \dots, n\}$, let j_i be the (unique) element of $\{1, \dots, n\}$ such that $\mathbf{a}_i j_i \in M$. We now add the row $\begin{bmatrix} j_1 & \dots & j_n \end{bmatrix}$ to the bottom of L, and we thus obtain an $(r+1) \times n$ Latin rectangle, which is what we needed. □

3 Vertex and edge connectivity

For a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, an *A-B path* in G, or a *path from A to B* in G, is either a one-vertex path whose sole vertex is in $A \cap B$, or a path on at least two vertices whose one endpoint is in A and whose other endpoint is in B.

Given a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, we say that a set $X \subseteq V(G)$ *separates* A from B in G if every path from A to B in G contains at least one vertex of X. Note that this implies that $A \cap B \subseteq X$.[13]

Given a graph G and a non-negative integer k, we say that G is *k-vertex-connected*, or simply *k-connected*, if $|V(G)| \geq k+1$ and for all $X \subseteq V(G)$ such that $|X| \leq k-1$, we have that $G \setminus X$ is connected. Note that this means that every (non-null) graph is 0-connected, and that every connected graph on at least two vertices is 1-connected.[14] The *connectivity* of a graph G, denoted $\kappa(G)$, is the largest integer k such that G is k-connected. Note that if $k = \kappa(G)$, then either $G = K_{k+1}$ or there exists a set of k vertices whose deletion from G yields a disconnected graph. Furthermore, if there exists a set of at most k vertices whose deletion from G yields a disconnected graph, then $\kappa(G) \leq k$.

Given a graph G and disjoint sets $A, B \subseteq V(G)$, we say that a set $F \subseteq E(G)$ *separates* A from B in G if every path from A to B contains at least one edge of F.

Given a graph G and a non-negative integer ℓ, we say that G is *ℓ-edge-connected* if $|V(G)| \geq 2$ and for all $F \subseteq E(G)$ such that $|F| \leq \ell - 1$, we have that $G \setminus F$ is connected. The *edge-connectivity* of a graph G on at least two vertices, denoted by $\lambda(G)$, is the largest integer ℓ such that G is

[13]Indeed, if $x \in A \cap B$, then x counts as a one-vertex path from A to B. So, any set of vertices that separates A from B must include $A \cap B$ as a subset.

[14]However, K_1 is **not** 1-connected.

ℓ-edge-connected. Note that if $\ell = \lambda(G)$, then there exists a set of ℓ edges whose deletion from G yields a disconnected graph. Furthermore, if there exists a set of at most ℓ edges whose deletion from G yields a disconnected graph, then $\lambda(G) \leq \ell$.

Proposition 3.1. *Let G be a graph on at least two vertices. Then*

(a) for all edges $e \in E(G)$, $\lambda(G) - 1 \leq \lambda(G \setminus e) \leq \lambda(G)$;

(b) for all sets $F \subseteq E(G)$, $\lambda(G \setminus F) \leq \lambda(G)$.

Proof. Clearly, (b) follows from (a) by an easy induction. It remains to prove (a). Fix $e \in E(G)$.

We first show that $\lambda(G \setminus e) \geq \lambda(G) - 1$. Fix $F \subseteq E(G \setminus e)$ such that $|F| \leq \lambda(G) - 2$. Set $F' = F \cup \{e\}$; then $|F'| \leq \lambda(G) - 1$, and we deduce that $G \setminus F'$ is connected. But $(G \setminus e) \setminus F = G \setminus F'$, and we deduce that $(G \setminus e) \setminus F$ is connected. This proves that $\lambda(G \setminus e) \geq \lambda(G) - 1$.

It remains to show that $\lambda(G \setminus e) \leq \lambda(G)$. Fix $F \subseteq E(G)$ with $|F| = \lambda(G)$, such that $G \setminus F$ is disconnected. Set $F' = F \setminus \{e\}$; then $|F'| \leq \lambda(G)$. Furthermore, we have that $(G \setminus e) \setminus F' = G \setminus F$, and we deduce that $(G \setminus e) \setminus F'$ is disconnected. Since $|F'| \leq \lambda(G)$, we see that $\lambda(G \setminus e) \leq \lambda(G)$. □

Proposition 3.2. *Let G be a graph on at least two vertices. Then*

(a) for all edges $e \in E(G)$, $\kappa(G) - 1 \leq \kappa(G \setminus e) \leq \kappa(G)$;

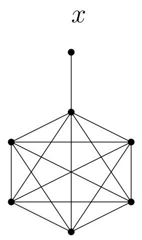
(b) for all sets $F \subseteq E(G)$, $\kappa(G \setminus F) \leq \kappa(G)$.

Proof. Clearly, (b) follows from (a) by an easy induction. It remains to prove (a). Fix $e \in E(G)$.

We first show that $\kappa(G \setminus e) \geq \kappa(G) - 1$. Since G is $\kappa(G)$-connected, we know that G (and consequently, $G \setminus e$ as well) has at least $\kappa(G) + 1$ vertices. Now, fix $X \subseteq V(G)$ such that $|X| \leq \kappa(G) - 2$; we must show that $(G \setminus e) \setminus X$ is connected. Suppose first that e is incident with some vertex in X. Then $(G \setminus e) \setminus X = G \setminus X$. Since $|X| \leq \kappa(G) - 2$, we see that $G \setminus X$ is connected, and it follows that $(G \setminus e) \setminus X$ is connected. It remains to consider the case when e is not incident with any vertex in X. Set $e = x_1x_2$ (i.e. let x_1 and x_2 be the endpoints of e). Set $X_1 := X \cup \{x_1\}$ and $X_2 := X \cup \{x_2\}$. Then $|X_1| = |X_2| = \kappa(G) - 1$, and we deduce that $G \setminus X_1$ and $G \setminus X_2$ are connected. Now, since $x_2 \in V(G) \setminus X_1$, and since $G \setminus X_1$ is a connected graph on at least two vertices, we see that x_2 is adjacent to some vertex in $u \in V(G) \setminus X_1$; since $x_1 \in X_1$, we see that $u \neq x_1$. Now, $(G \setminus e) \setminus X$ can be obtained from the connected graph $G \setminus X_2$ by adding to it the vertex x_2 and making it adjacent to all vertices in $N_G(x_2) \setminus \{x_1\}$. Since $u \in N_G(x_2) \setminus \{x_1\}$, we see that x_2 is not an isolated vertex of $(G \setminus e) \setminus X$, and we deduce that $(G \setminus e) \setminus X$ is connected. This proves that $\kappa(G \setminus e) \geq \kappa(G) - 1$.

It remains to show that $\kappa(G \setminus e) \leq \kappa(G)$. By definition, $|V(G)| \geq \kappa(G)+1$. If G has precisely $\kappa(G)+1$ vertices, then so does $G \setminus e$, and it follows from the definition that $\kappa(G \setminus e) \leq \kappa(G)$. It remains to consider the case when $|V(G)| \geq \kappa(G)+2$. In this case, there exists a set $X \subseteq V(G)$ of size $\kappa(G)$ such that $G \setminus X$ is disconnected. But then $(G \setminus e) \setminus X$ is disconnected as well, and it follows that $\kappa(G \setminus e) \leq \kappa(G)$. □

We note that, unlike edge deletion, vertex deletion sometimes increases connectivity. For instance, for the graph G represented below, we have that $\kappa(G) = \lambda(G) = 1$, but $\kappa(G \setminus x) = \lambda(G \setminus x) = 5$.



G

Recall that for a graph G, $\delta(G)$ is the minimum and $\Delta(G)$ the maximum degree in G, i.e. $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ and $\Delta(G) = \max\{d_G(v) \mid v \in V(G)\}$.

Theorem 3.3. *Let G be a graph on at least two vertices. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Proof. We first prove that $\lambda(G) \leq \delta(G)$. Fix a vertex $v \in V(G)$ such that $d_G(v) = \delta(G)$, and let F be the set of all edges of G that are incident with v. Clearly, $G \setminus F$ is disconnected, and it follows that $\lambda(G) \leq \delta(G)$.

It remains to show that $\kappa(G) \leq \lambda(G)$. Fix a set $F \subseteq E(G)$ such that $|F| = \lambda(G)$ and $G \setminus F$ is disconnected.

> **Claim.** If C is the vertex set of a component of $G \setminus F$, then no edge of F has both its endpoints in C.

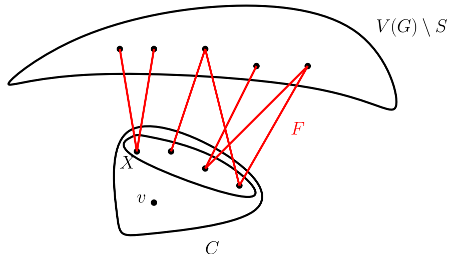
Proof of the Claim. Suppose otherwise. Let C be the vertex set of a component of $G \setminus F$,[15] and let $e \in F$ be an edge that has both its endpoints in C. Then $G \setminus (F \setminus \{e\})$ is still disconnected,[16] contrary to the fact that $|F \setminus \{e\}| = |F| - 1 = \lambda(G) - 1$. This proves the Claim. ■

Suppose first that there exists a vertex $v \in V(G)$ that is not incident with any edge in F. Let C be the vertex set of the component of $G \setminus F$ that

[15]Since $G \setminus F$ is disconnected, this implies that C and $V(G) \setminus C$ are both non-empty, and there are no edges between them.

[16]This is because there are still no edges between C and $V(G) \setminus C$, and both C and $V(G) \setminus C$ are non-empty.

contains v. By the Claim, no edge in F has both endpoints in C. Now, let X be the set of all vertices in C that are incident with an edge in F. Then $|X| \leq |F| = \lambda(G)$ and $G \setminus X$ is disconnected. So, $\kappa(G) \leq \lambda(G)$.



It remains to consider the case when every vertex of G is incident with an edge of F.[17] Fix any $v \in V(G)$; we will show that $d_G(v) \leq \lambda(G)$. Let C be the component of $G \setminus F$ that contains v, and let F_v be the set of edges of F incident with v. Let $u_1, \dots, u_t$ be the neighbors of v in the component C, and for all $i \in \{1, \dots, t\}$, let F_i be the set of all edges of F incident with u_i. By supposition, sets $F_v, F_1, \dots, F_t$ are all non-empty, and by the Claim, they are pairwise disjoint. So,

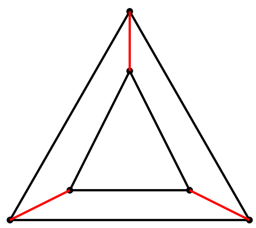
$$d_G(v) \quad = \quad |F_v| + t \quad \leq \quad |F_v| + |F_1| + \dots + |F_t| \quad \leq \quad |F| \quad = \quad \lambda(G),$$

as we had claimed. Since we chose v arbitrarily, it now follows that $\Delta(G) \leq \lambda(G)$; we already saw that $\lambda(G) \leq \delta(G)$, and we now deduce that $\lambda(G) = \Delta(G)$. Now, if G is a complete graph, then $|V(G)| = \Delta(G) + 1$, and we see that $\kappa(G) = \Delta(G) = \lambda(G)$. So assume that G is not complete, and fix some $x \in V(G)$ that has a non-neighbor in G. Then $G \setminus N_G(x)$ is disconnected, and we have that $|N_G(x)| = d_G(x) \leq \Delta(G) = \lambda(G)$. So, $\kappa(G) \leq \lambda(G)$. $\square$

Terminology: A *vertex-cutset* of a graph G is any set $X \subsetneqq V(G)$ such that $G \setminus X$ has more components than G.[18] Similarly, an *edge-cutset* of G is any set $F \subseteq E(G)$ such that $G \setminus F$ has more components than G.

By definition, no graph G has a vertex-cutset of size strictly smaller than $\kappa(G)$. Similarly, no graph G has an edge-cutset of size strictly smaller than $\lambda(G)$.

[17] For an example, see the graph below, with the edges of F in red.



[18] So, if G is connected, then a vertex-cutset of G is any set $X \subsetneqq V(G)$ such that $G \setminus X$ is disconnected.