NDMI011: Combinatorics and Graph Theory 1

Lecture #4

Finite projective planes (part I)

Irena Penev

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Definition

A *finite projective plane* is set system $(X, \mathcal{P})$ s.t. X is a finite, and the following three properties are satisfied:

(P0) there exists a 4-element subset $Q \subseteq X$ s.t. every $P \in \mathcal{P}$ satisfies $|P \cap Q| \leq 2$;

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- For distinct points $a, b \in X$, we denote by $\overline{ab}$ the unique line in $\mathcal{P}$ that contains a and b (the existence and uniqueness of such a line follow from (P2)).

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- (P2) is the same as for points and lines in the Euclidean plane.
- But (P1) is different! There are no "parallel lines" in a finite projective plane.

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Example 1.1

Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{P} = \{a, b, c, d, e, f, g\}$, where

- $a = \{1, 2, 3\}$,
- $b = \{3, 4, 5\}$,
- $c = \{5, 6, 1\}$,
- $d = \{5, 7, 2\}$,
- $e = \{1, 7, 4\}$,
- $f = \{3, 7, 6\}$,
- $g = \{2, 4, 6\}$.

Then $(X, \mathcal{P})$ is a finite projective plane,[a] called the *Fano plane*.

[a] It is easy to check that (P1) and (P2) are satisfied. For (P0), we can take, for instance, $Q = \{1, 3, 5, 7\}$.

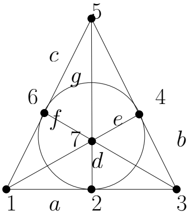


Figure: The Fano plane.

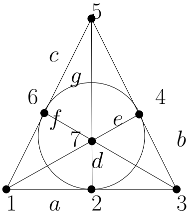
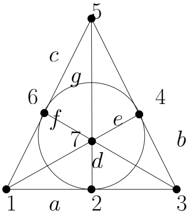


Figure: The Fano plane.

- In the picture above, the seven lines of the Fano plane are represented by six line segments and one circle.



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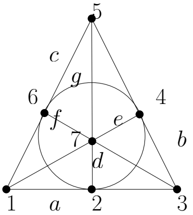

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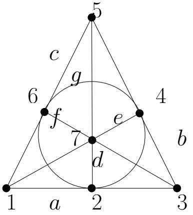
- In the picture above, the seven lines of the Fano plane are represented by six line segments and one circle.
- However, formally, each line of the Fano plane is simply a set of three points.
- Drawings can sometimes be useful for guiding our intuition. However, formal proofs should never rely on such pictures; instead, they should rely solely on the definition of a finite projective plane or on results (propositions, lemmas, theorems) proven about them.

- The *incidence graph* of a finite projective plane $(X, \mathcal{P})$ is a bipartite graph with bipartition $(X, \mathcal{P})$, in which $x \in X$ and $P \in \mathcal{P}$ are adjacent if and only if $x \in P$.



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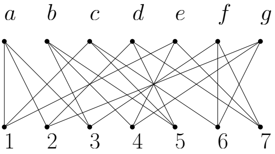
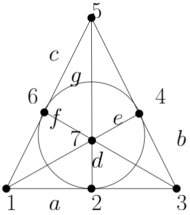


Figure: The incidence graph of the Fano plane.



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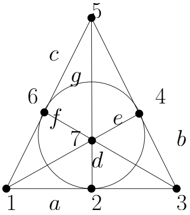

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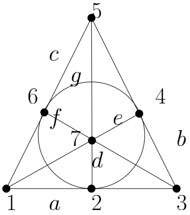

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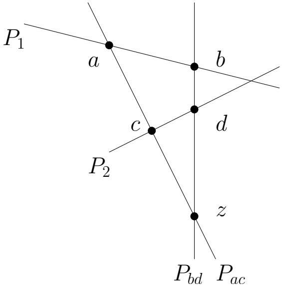
Proof of the Claim (outline, continued).



This proves the Claim.

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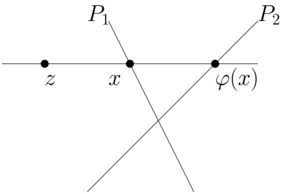

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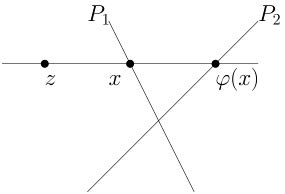

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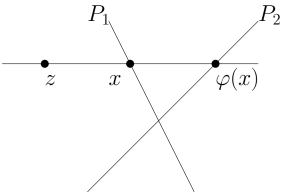

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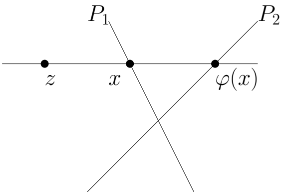

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[a] So, if $(X, \mathcal{P})$ is a finite projective plane of order n, then each line in $\mathcal{P}$ contains exactly $n + 1$ points.

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- The Fano plane has order two.

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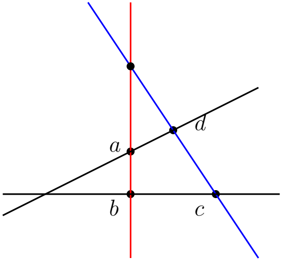
Let $Q = \{a, b, c, d\}$ satisfy (P0). Then the line $\overline{ab}$ has at least three points (namely, a, b, and its point of intersection with $\overline{cd}$).



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Let $(X, \mathcal{P})$ be a finite projective plane of order n. Then all the following hold:

(a) for each point $x \in X$, exactly $n+1$ lines in $\mathcal{P}$ pass through x;

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- We prove (c) after introducing "duality" (we use (a) and (b)).

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Claim. *For every point $x \in X$, there exists a line $P \in \mathcal{P}$ s.t. $x \notin P$.*

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Let $(X, \mathcal{P})$ be a finite projective plane of order n. Then all the following hold:

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Proof of the Claim (outline). Fix a point $x \in X$. Using (P0) from the definition of a finite projective plane, we fix a 4-element subset $Q \subseteq X$ s.t. for all $P \in \mathcal{P}$, $|Q \cap P| \leq 2$.

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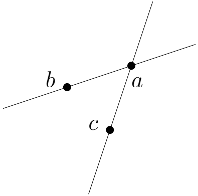
Claim. *For every point $x \in X$, there exists a line $P \in \mathcal{P}$ s.t. $x \notin P$.*

Proof of the Claim (outline, continued).



Then x belongs to at most one of $\overline{ab}$ and $\overline{ac}$. This proves the Claim.

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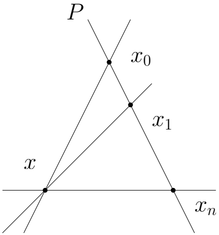
Proof of (a) (outline). Fix a point $x \in X$. By the Claim, there exists a line $P \in \mathcal{P}$ s.t. $x \notin P$. Since $(X, \mathcal{P})$ is of order n, we know that $|P| = n + 1$; set $P = \{x_0, x_1, \ldots, x_n\}$.



At least $n+1$ lines (namely, $\overline{xx_0}, \ldots, \overline{xx_n}$) pass through x.

Every line through x intersects P, so $\overline{xx_0}, \ldots, \overline{xx_n}$ are the only lines through x.

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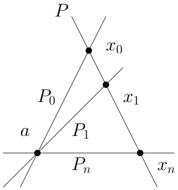
Proof of (b) (outline). Fix any line $P \in \mathcal{P}$. Since $(X, \mathcal{P})$ is of order n, we know that $|P| = n+1$; set $P = \{x_0, x_1, \dots, x_n\}$. Since every line in $\mathcal{P}$ has $n+1$ points, the Claim guarantees that $|X| \geq n+2$; consequently, $P \subsetneqq X$. Fix any $a \in X \setminus P$.



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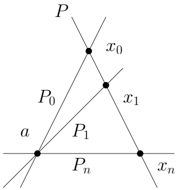

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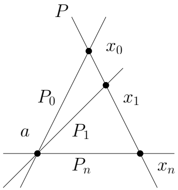

By (P1) and (P2), $P_i \cap P_j = \{a\}$ for all distinct $i, j \in \{0, 1, \ldots, n\}$; consequently, $P_0 \setminus \{a\}, P_1 \setminus \{a\}, \ldots, P_n \setminus \{a\}$ are pairwise disjoint.

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Proof of (b) (outline, continued). Consequently,

$$\begin{aligned} & |P_0 \cup P_1 \cup \cdots \cup P_n| \\ = \; & |\{a\}| + |P_0 \setminus \{a\}| + |P_1 \setminus \{a\}| + \cdots + |P_n \setminus \{a\}| \\ = \; & 1 + (n+1)n \\ = \; & n^2 + n + 1. \end{aligned}$$

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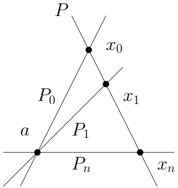
It now remains to show that $X = P_0 \cup P_1 \cup \cdots \cup P_n$; in fact, we only need to show that $X \subseteq P_0 \cup P_1 \cup \cdots \cup P_n$, for the reverse inclusion is immediate.

Proof of (b) (outline, continued). Reminder: WTS $X \subseteq P_0 \cup P_1 \cup \cdots \cup P_n$.



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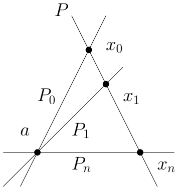

Fix a point $x \in X$; WTS x belongs to at least one of $P_0, P_1, \ldots, P_n$.

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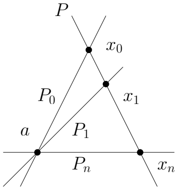

Fix a point $x \in X$; WTS x belongs to at least one of $P_0, P_1, \ldots, P_n$. WMA $x \neq a$, for otherwise we are done.

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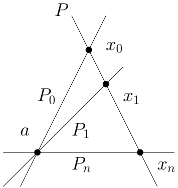

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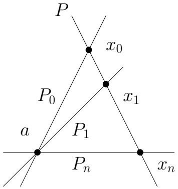

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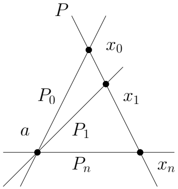

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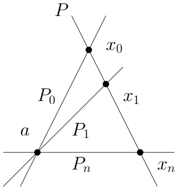

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Definition

For a set system $(X, \mathcal{S})$, we define the *dual* of $(X, \mathcal{S})$ to be the ordered pair $(Y, \mathcal{T})$, where $Y = \mathcal{S}$ and

$$\mathcal{T} = \Big\{ \{ S \in \mathcal{S} \mid x \in S \} \ \Big| \ x \in X \Big\}.$$

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Example 2.1

Let $X = \{1, 2, 3\}$ and $\mathcal{S} = \{A, B\}$, where $A = \{1, 2\}$ and $B = \{1, 3\}$. Then the dual of $(X, \mathcal{S})$ is $(Y, \mathcal{T})$, where $Y = \{A, B\}$ and $\mathcal{T} = \Big\{\{A, B\}, \{A\}, \{B\}\Big\}$.[a]

[a] Indeed $\{S \in \mathcal{S} \mid 1 \in S\} = \{A, B\}$, $\{S \in \mathcal{S} \mid 2 \in S\} = \{A\}$, and $\{S \in \mathcal{S} \mid 3 \in S\} = \{B\}$.

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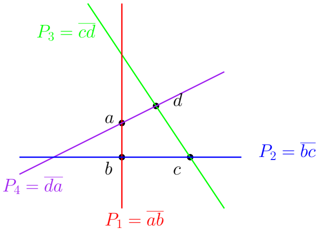
Proof of (P0) for $(Y, \mathcal{R})$ (outline): Let $Q = \{a, b, c, d\}$ be as in (P0) for $(X, \mathcal{P})$.



Then $Q^* = \{P_1, P_2, P_3, P_4\}$ satisfies (P0) for the dual $(Y, \mathcal{R})$.

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Proof of (P2) for $(Y, \mathcal{R})$ (outline): For all distinct $P_1, P_2 \in Y = \mathcal{P}$, (P1) for $(X, \mathcal{P})$ guarantees that $|P_1 \cap P_2| = 1$, say $P_1 \cap P_2 = \{x\}$;

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Proof of (P2) for $(Y, \mathcal{R})$ (outline): For all distinct $P_1, P_2 \in Y = \mathcal{P}$, (P1) for $(X, \mathcal{P})$ guarantees that $|P_1 \cap P_2| = 1$, say $P_1 \cap P_2 = \{x\}$; then R_x is the unique line of $(Y, \mathcal{R})$ containing both P_1 and P_2.

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Proof of (c).

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Proof of (c). By Theorem 2.2, the dual $(Y, \mathcal{R})$ of $(X, \mathcal{P})$ is a finite projective plane. We have $Y = \mathcal{P}$ and $\mathcal{R} = \{R_x \mid x \in X\}$, where $R_x = \{P \in \mathcal{P} \mid x \in P\}$ for all $x \in X$.

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Proof of (c). By Theorem 2.2, the dual $(Y, \mathcal{R})$ of $(X, \mathcal{P})$ is a finite projective plane. We have $Y = \mathcal{P}$ and $\mathcal{R} = \{R_x \mid x \in X\}$, where $R_x = \{P \in \mathcal{P} \mid x \in P\}$ for all $x \in X$. By (a), each R_x contains exactly $n+1$ members of $\mathcal{P}$.

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Proof of (c). By Theorem 2.2, the dual $(Y, \mathcal{R})$ of $(X, \mathcal{P})$ is a finite projective plane. We have $Y = \mathcal{P}$ and $\mathcal{R} = \{R_x \mid x \in X\}$, where $R_x = \{P \in \mathcal{P} \mid x \in P\}$ for all $x \in X$. By (a), each R_x contains exactly $n+1$ members of $\mathcal{P}$. So, the order of $(Y, \mathcal{R})$ is n.

Theorem 2.2

The dual of a finite projective plane is again a finite projective plane.

Theorem 1.4

Let $(X, \mathcal{P})$ be a finite projective plane of order n. Then all the following hold:

(a) for each point $x \in X$, exactly $n+1$ lines in $\mathcal{P}$ pass through x;

(b) $|X| = n^2 + n + 1$;

(c) $|\mathcal{P}| = n^2 + n + 1$.

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