NDMI011: Combinatorics and Graph Theory 1

Lecture #3

Generating functions (part II)

Irena Penev

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This lecture consists of three parts:

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1. Basic operations with generating functions;

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2. Application #1: counting binary trees;

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1. Basic operations with generating functions;
2. Application #1: counting binary trees;
3. Application #2: random walks.

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Suppose $\{a_n\}_{n=0}^{\infty}$ is some infinite sequence of real (or complex) numbers. The *generating function* of this sequence is the power series

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- Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences with corresponding generating functions $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n$, and let α be a constant.

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  (1) The generating function of the sequence $\{a_n + b_n\}_{n=0}^{\infty}$ is $a(x) + b(x)$.
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  (5) For an integer $k \geq 1$, the generating function of the sequence $\{a_{n+k}\}_{n=0}^{\infty}$, i.e. the sequence $a_k, a_{k+1}, a_{k+2}, \ldots,$ is $\frac{1}{x^k}\left(a(x) - \sum_{i=0}^{k-1} a_i x^i\right)$.

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    - For instance, since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is the generating function of $1, 1, 1, 1, 1, \ldots,$ we see that $\frac{1}{1-2x}$ $(= \sum_{n=0}^{\infty} 2^n x^n)$ is the generating function of $1, 2, 4, 8, 16, \ldots$.

- Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences with corresponding generating functions $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n$, and let α be a constant.

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  (7) For an integer $k \geq 1$, the generating function of the sequence

$$a_0, \underbrace{0, \ldots, 0}_{k}, a_1, \underbrace{0, \ldots, 0}_{k}, a_2, \underbrace{0, \ldots, 0}_{k}, a_3, \ldots$$

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  is $a(x^{k+1})$.
    - For instance, the generating function of the sequence $a_0, 0, 0, a_1, 0, 0, a_2, 0, 0, a_3, \ldots$ is $a(x^3)$ $(= \sum_{n=0}^{\infty} a_n x^{3n})$.

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  (8) The generating function of the sequence $\{(n+1)a_{n+1}\}_{n=0}^{\infty}$, i.e. the sequence $a_1, 2a_2, 3a_3, 4a_4, \ldots$, is $a'(x)$.

  The generating function for the sequence $0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \frac{1}{4}a_3, \ldots$ is $\int_0^x a(t)dt$.

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  (We differentiate and integrate power series term-by-term.)
  (9) The function $c(x) = a(x)b(x)$ is the generating function of the sequence $\{c_n\}_{n=0}^{\infty}$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for each integer $n \geq 0$.
    - So, $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b_0$, $c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$, etc.

Reminder: For a sequence $\{a_n\}_{n=0}^{\infty}$ with generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and a constant α:

(6) The generating function of the sequence $\{\alpha^n a_n\}_{n=0}^{\infty}$ is $c(x) = a(\alpha x)$.

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Let $\{a_n\}_{n=0}^{\infty}$ be a sequence, and let $a(x)$ be its generating function. Find the generating function of the sequence $a_0, 0, a_2, 0, a_4, \ldots$ in terms of the function $a(x)$.

Solution. $a_0, 0, a_2, 0, a_4, \ldots$ is the sum of $\{\frac{a_n}{2}\}_{n=0}^{\infty}$ and $\{\frac{(-1)^n a_n}{2}\}_{n=0}^{\infty}$.

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Example 1.2

Find (the closed form of) the generating function of the sequence $1, 1, 2, 2, 4, 4, 8, 8, 16, 16, \ldots$, i.e. the sequence $\{2^{\lfloor n/2 \rfloor}\}_{n=0}^{\infty}$.

Solution.

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Find (the closed form of) the generating function of the sequence $1, 1, 2, 2, 4, 4, 8, 8, 16, 16, \ldots$, i.e. the sequence $\{2^{\lfloor n/2 \rfloor}\}_{n=0}^{\infty}$.

Solution. Recall that the generating function of the sequence $1, 2, 4, 8, 16, \ldots$ is $\frac{1}{1-2x}$.

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Find (the closed form of) the generating function of the sequence $1^2, 2^2, 3^2, 4^2, \ldots$, i.e. the sequence $\{(n+1)^2\}_{n=0}^{\infty}$.

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Solution. The generating function of the sequence $1, 1, 1, 1, \ldots$ is $\frac{1}{1-x}$. By differentiating, we see that $\frac{d}{dx}(\frac{1}{1-x}) = \frac{1}{(1-x)^2}$ is the generating function of the sequence $1, 2, 3, 4, \ldots$, i.e. the sequence $\{n+1\}_{n=0}^{\infty}$.

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$$a(x) = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}.$$

Part II: Application #1: counting binary trees

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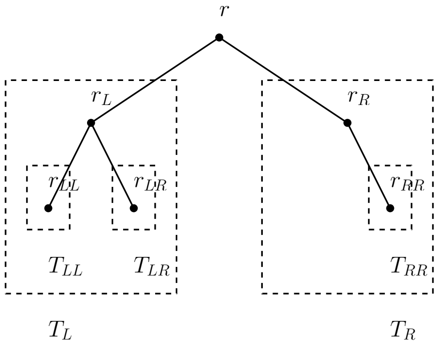
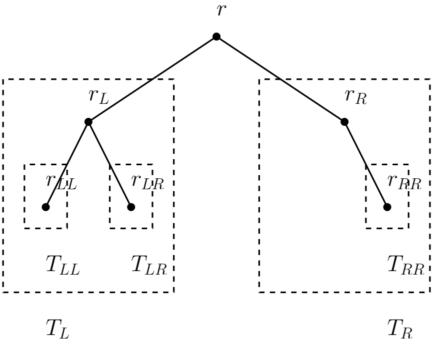
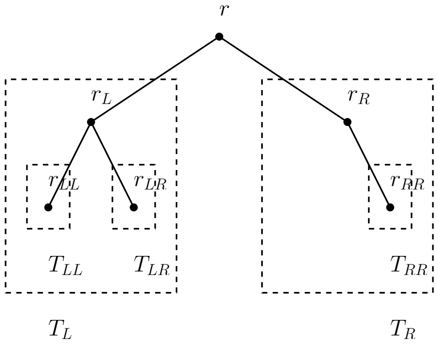
- We define binary trees recursively as follows: a *binary tree* is either empty (i.e. contains no nodes), or consists of designated node r (called the *root*), plus an ordered pair (T_L, T_R) of binary trees, where T_L and T_R (called the *left subtree* and the *right subtree*) have disjoint sets of nodes and do not contain the node r.





- Remark: The empty binary tree has zero nodes, and if a binary tree T consists of a root r and an ordered pair (T_L, T_R) of binary trees, then the number of nodes of T is $1 + n_L + n_R$, where n_L is the number of nodes of T_L, and n_R is the number of nodes of T_R.



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- Goal: Count the number of binary trees on n nodes ($n \geq 0$).

- For each integer $n \geq 0$, let b_n be the number of binary trees on n nodes, and let $b(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating function of the sequence $\{b_n\}_{n=0}^{\infty}$.

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- It is easy to check that $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, and $b_3 = 5$.

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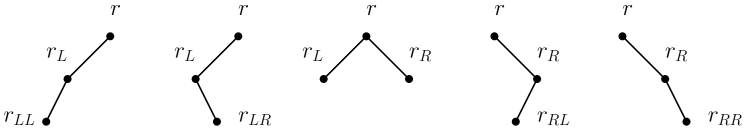
- It is easy to check that $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, and $b_3 = 5$.
- Here are all the binary trees on three nodes:



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- The number of binary trees on $n \geq 1$ nodes is equal to the number of ordered pairs (T_L, T_R) of binary trees s.t. T_L, T_R together have $n-1$ nodes.

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- The number of binary trees on $n \geq 1$ nodes is equal to the number of ordered pairs (T_L, T_R) of binary trees s.t. T_L, T_R together have $n-1$ nodes.
- Thus, for all integers $n \geq 1$, we have that

$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + \cdots + b_{n-1} b_0 = \sum_{k=0}^{n-1} b_k b_{n-k-1}.$$

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$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + \cdots + b_{n-1} b_0 = \sum_{k=0}^{n-1} b_k b_{n-k-1}.$$

- Since $b_0 = 1$, this implies that $b(x) = 1 + x b(x)^2$.

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- Let's find a recursive formula for b_n ($n \geq 1$).
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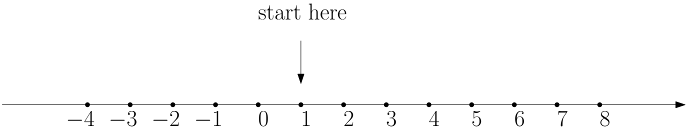

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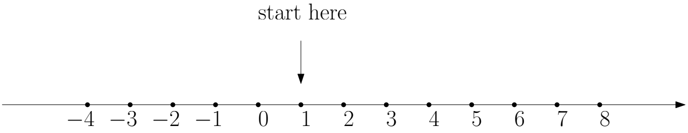

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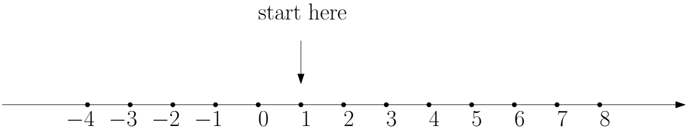

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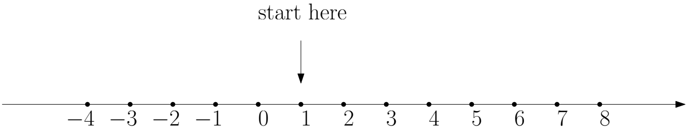

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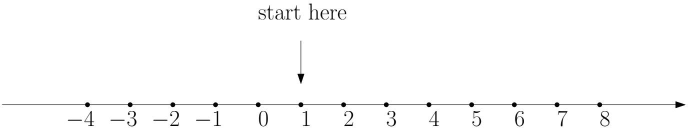

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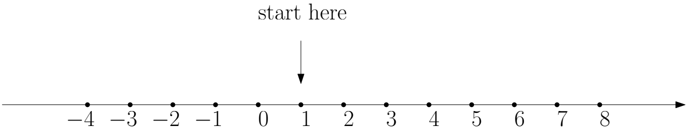

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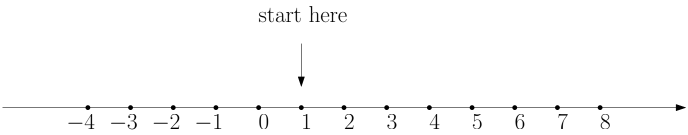
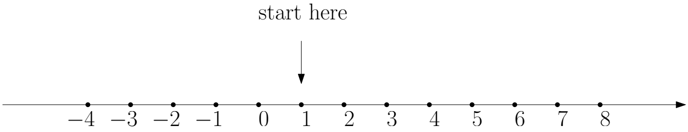

start here
−4 −3 −2 −1 0 1 2 3 4 5 6 7 8



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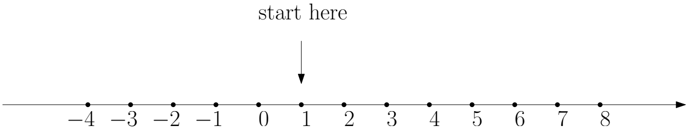

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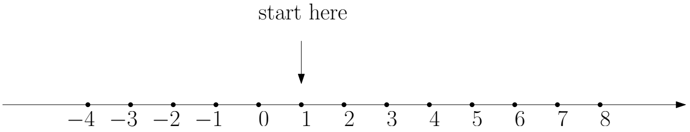

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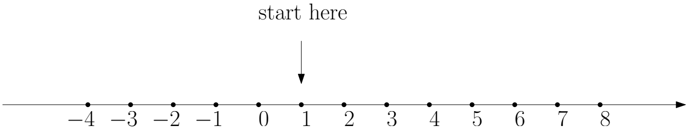

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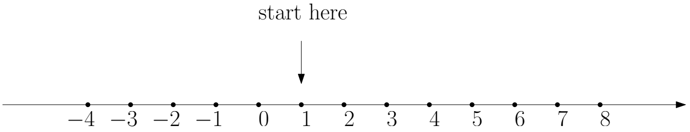

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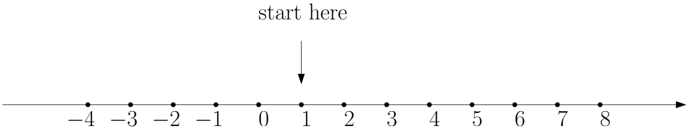

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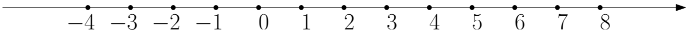

- For our solution, it will be useful to consider random walks that start at points other than 1, but still proceed according to the same rules:
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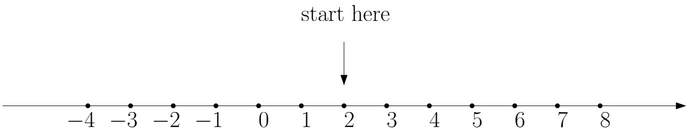

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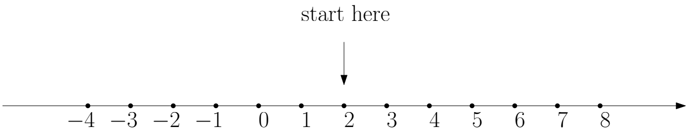

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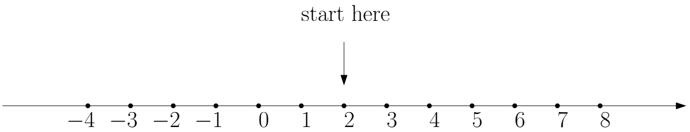

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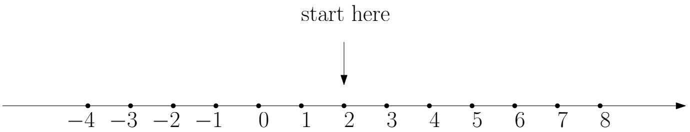

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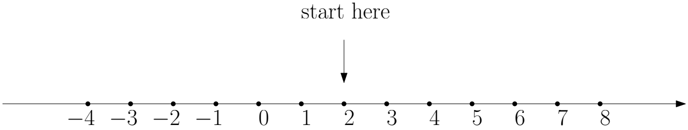

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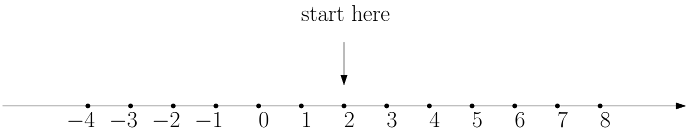

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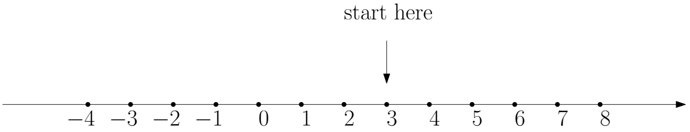

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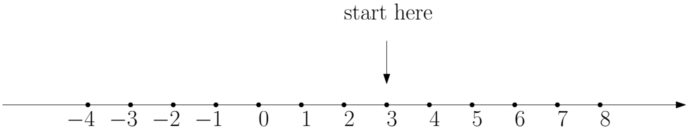

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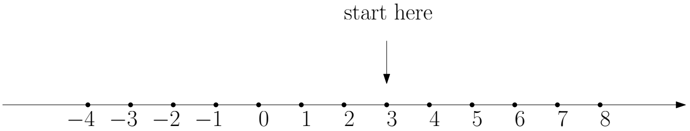

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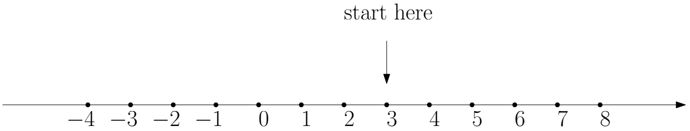

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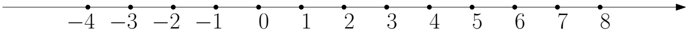
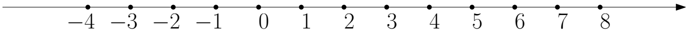
- Since $b(x) = a(x)^2$, we get

$$c(x) = a(x)^3.$$



- Reminder: For an integer $n \geq 0$:
  - a_n is the number of ways to reach the origin for the first time after precisely n steps, starting from 1.
  - c_n is the number of ways to reach the origin for the first time after precisely n steps, starting from 3.



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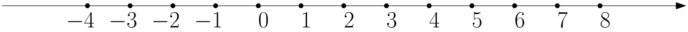

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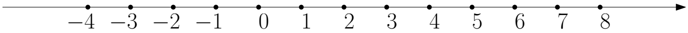
- If we start at 1, then for an integer $n \geq 2$, there are precisely c_{n-1} ways to reach the origin for the first time after precisely n steps: we must first move two units to the right, and then reach the origin from 3 for the first time after precisely $n - 1$ steps.
  - Thus, $a_n = c_{n-1}$ for all integers $n \geq 2$.



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- We now compute...

$$
\begin{aligned}
a(x) &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= x + x \sum_{n=2}^{\infty} a_n x^{n-1} && \text{because } a_0 = 0 \text{ and } a_1 = 1 \\
&= x + x \sum_{n=2}^{\infty} c_{n-1} x^{n-1} && \text{because } a_n = c_{n-1} \text{ for } n \geq 2 \\
&= x + x \sum_{n=1}^{\infty} c_n x^n \\
&= x + x \sum_{n=0}^{\infty} c_n x^n && \text{because } c_0 = 0 \text{ (obvious)} \\
&= x + x c(x).
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- Let's show that $P \neq 1$, so that $P = \varphi$.

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- So, $P = \varphi$, i.e.
$$P = \frac{-1+\sqrt{5}}{2}.$$
- Thus, the probability that we ever reach the origin in our walk is $\frac{-1+\sqrt{5}}{2}$ (the golden ratio).