NDMI011: Combinatorics and Graph Theory 1

Lecture #2

Generating functions (part I)

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1. Partial fractions;

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1. Partial fractions;
2. A review of Taylor (and Maclaurin) series;
3. An introduction to generating functions.

Part I: Partial fractions

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- So, we write

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

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In general, suppose $p(x)$ and $q(x)$ are polynomials with complex coefficients such that $\deg p(x) < \deg q(x)$, and such that

$$q(x) = c(x - \alpha_1)^{\beta_1} \dots (x - \alpha_t)^{\beta_t},$$

where c is a non-zero complex number, $\alpha_1, \dots, \alpha_t$ are pairwise distinct complex numbers, and $\beta_1, \dots, \beta_t$ are positive integers.

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$$\frac{p(x)}{q(x)} = \frac{A_{1,1}}{x - \alpha_1} + \dots + \frac{A_{1,\beta_1}}{(x - \alpha_1)^{\beta_1}} + \dots + \frac{A_{t,1}}{x - \alpha_t} + \dots + \frac{A_{t,\beta_t}}{(x - \alpha_t)^{\beta_t}}.$$

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Finding $A_{1,1}, \dots, A_{1,\beta_1}, \dots, A_{t,1}, \dots, A_{t,\beta_t}$ reduces to solving a system of linear equations, as in the example that we considered.

- For example:

$$\frac{x^5-7x+1}{7(x-2)^3(x+1)^2(x+2)^4} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{D}{x+1}+\frac{E}{(x+1)^2}+$$
$$+\frac{F}{x+2}+\frac{G}{(x+2)^2}+\frac{H}{(x+2)^3}+\frac{I}{(x+2)^4}.$$

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- See the Lecture Notes for another fully worked out example.

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- Then we first perform polynomial division, and then we perform our procedure on the remainder.
- For instance:

$$\begin{aligned} \frac{3x^4-3x^3+1}{x^2(x-1)} &= 3x + \frac{1}{x^2(x-1)} \\ &= 3x - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} \end{aligned}$$

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Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in A$, and assume that A contains (as a subset) some open neighborhood of a, and that f is infinitely differentiable at a. Then the *Taylor series* of f centered at a is the series

$$T^{f,a}(x) \quad = \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

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- The Taylor series $T^{f,0}(x)$ (here, we have $a = 0$) is called the *Maclaurin series*.

Here are the Maclaurin series of some familiar functions (from analysis):

(i) $T^{\exp(x),0}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \ldots;$

(ii) $T^{\sin x,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots;$

(iii) $T^{\cos x,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n}\frac{x^{2n}}{(2n)!} + \ldots;$

(iv) $T^{\ln(1+x),0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^{n}}{n} + \ldots;$

(v) $T^{(1+x)^\alpha,0}(x) = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \ldots$, where α is a fixed real number;

(vi) $T^{\frac{1}{1-x},0}(x) = 1 + x + x^2 + \cdots + x^n + \ldots.$

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5. (v) $T^{(1+x)^\alpha,0}(x) = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \ldots$, where α is a fixed real number;
6. (vi) $T^{\frac{1}{1-x},0}(x) = 1 + x + x^2 + \cdots + x^n + \ldots.$

- Let's verify (v).
- Actually, what does $\binom{\alpha}{k}$ mean when α is a **real** number?

Definition

For a real number α and a non-negative integer k, we define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}.$$

In particular, $\binom{\alpha}{0} = 1$.

- (v) $T^{(1+x)^\alpha,0}(x) = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \ldots$, where α is a fixed real number.

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- By induction, for all integers $k \geq 0$:

$$\frac{d^k}{dx^k}(1+x)^\alpha = \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k}.$$

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- So,

$$\frac{\frac{d^k}{dx^k}(1+x)^\alpha\Big|_{x=0}}{k!} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} = \binom{\alpha}{k}.$$

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- And now (v) follows.

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- Even if does converge, it need not converge to $f(x)$.
- Nevertheless, we have the following:

1. $\exp(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \ldots$ for all $x \in \mathbb{R}$;
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots$ for all $x \in \mathbb{R}$;
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{\frac{x^{2n}}{(2n)!}} + \ldots$ for all $x \in \mathbb{R}$;
4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \ldots$ for all $x \in (-1, 1]$;
5. $(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \ldots$ for $x \in (-1, 1)$, where α is a fixed real number;
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- (5) is called the "Generalized Binomial Theorem."

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- (5) is called the "Generalized Binomial Theorem."
- If α is a non-negative integer, then for integers $k > \alpha$, we have $\binom{\alpha}{k} = 0$, and so

$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \cdots + \binom{\alpha}{\alpha}x^\alpha,$$

which is what we also get via the (finite) Binomial Theorem.

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- For a constant $a \neq 0$, an integer $t \geq 1$, and a sufficiently small value of x, we can substitute ax^t for x in the above equations.

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- For a constant $a \neq 0$, an integer $t \geq 1$, and a sufficiently small value of x, we can substitute ax^t for x in the above equations.
- For example, by substituting $2x^3$ for x in (6), we get that

$$\frac{1}{1-2x^3} = 1 + 2x^3 + 4x^6 + \cdots + 2^n x^{3n} + \dots$$

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4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \dots$ for all $x \in (-1, 1]$;
5. $(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \dots$ for $x \in (-1, 1)$, where α is a fixed real number;
6. $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \dots$ for $x \in (-1, 1)$.

- For a constant $a \neq 0$, an integer $t \geq 1$, and a sufficiently small value of x, we can substitute ax^t for x in the above equations.
- For example, by substituting $2x^3$ for x in (6), we get that

$$\frac{1}{1-2x^3} = 1 + 2x^3 + 4x^6 + \cdots + 2^n x^{3n} + \dots$$

- (6) follows from (5), with $\alpha = -1$ and $-x$ substituted for x.

1. $\exp(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \dots$ for all $x \in \mathbb{R}$;
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- In working with generating functions, we will not worry about exactly how small x needs to be to make our equations work.
- We simply need that they work for values of x in some (no matter how small) open neighborhood of zero.

Part III: Generating functions

Part III: Generating functions

- Motivating example:

  How many ways are there to pay 21 *Kč, assuming we have six* 1 *Kč coins, five* 2 *Kč coins, and four* 5 *Kč coins?*

  (Here, we treat all coins of the same value as the same. So, if we happened to use three 1 Kč coins, we do not care which particular three we chose.)

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How many ways are there to pay 21 Kč, assuming we have six 1 Kč coins, five 2 Kč coins, and four 5 Kč coins?

- We are looking for the number of solutions to the equation

$$i_1 + i_2 + i_5 = 21,$$

with $i_1 \in \{0, 1, 2, 3, 4, 5, 6\}$, $i_2 \in \{0, 2, 4, 6, 8, 10\}$, and $i_5 \in \{0, 5, 10, 15, 20\}$.

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- This is precisely the coefficient in front of x^{21} in the following polynomial:

$$\begin{aligned} p(x) &= (1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \\ &\quad \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10}) \\ &\quad \times (1 + x^5 + x^{10} + x^{15} + x^{20}) \end{aligned}$$

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- Indeed, we obtain x^{21} by selecting some x^{i_1} from the first term of the product, some x^{i_2} from the second, and some x^{i_5} from the third, in such a way that $i_1 + i_2 + i_5 = 21$.

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- The number of ways of selecting i_1, i_2, i_5 is precisely the coefficient in front of x^{21} in the polynomial $p(x)$.

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- We call $p(x)$ the "generating function" of the sequence $\{a_n\}_{n=0}^{\infty}$.
- In this case, it is a polynomial, but in general, it is a (potentially infinite) series.

Definition

Suppose $\{a_n\}_{n=0}^{\infty}$ is some infinite sequence of real (or complex) numbers. The *generating function* of this sequence is the power series

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- An application of generating functions: difference equations.

Definition

For a positive integer k, a *homogeneous linear difference equation of degree k* is an equation of the form

$$y_{n+k} = a_{k-1}y_{n+k-1} + a_{k-2}y_{n+k-2} + \cdots + a_1 y_{n+1} + a_0 y_n,$$

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- One famous example of such a sequence is the *Fibonacci sequence* $\{F_n\}_{n=0}^{\infty}$, defined recursively as follows:
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- But often, this isn't so easy!
- What is a closed formula for the n-th Fibonacci number F_n??

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- However, in practice, if our difference equation is of high degree, this may be difficult or impossible to do due to problems with factoring polynomials of high degree.
- Let's show how this can be done for sequences defined via second degree homogeneous linear difference equations.
- We do this for the Fibonacci sequence (and there is one more worked out example in the Lecture Notes).

Example

Find a closed formula of the general term of the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$, defined recursively as follows:

- $F_0 = 0$, $F_1 = 1$;
- $F_{n+2} = F_n + F_{n+1}$ for all integers $n \geq 0$.

Solution.

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Solution. We consider the generating function

$$f(x) = \sum_{n=0}^{\infty} F_n x^n$$

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Solution. We consider the generating function

$$f(x) = \sum_{n=0}^{\infty} F_n x^n$$

for $\{F_n\}_{n=0}^{\infty}$. We manipulate the above series as follows:

Solution (continued). Reminder: $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_n + F_{n+1}\ \forall n \geq 0$.

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\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} F_n x^n \\
&= F_0 + F_1 x + x^2 \sum_{n=0}^{\infty} F_{n+2} x^n \\
&= x + x^2 \sum_{n=0}^{\infty} (F_n + F_{n+1}) x^n \\
&= x + (x^2 \sum_{n=0}^{\infty} F_n x^n) + (x^2 \sum_{n=0}^{\infty} F_{n+1} x^n) \\
&= x + (x^2 \sum_{n=0}^{\infty} F_n x^n) + (x \sum_{n=0}^{\infty} F_{n+1} x^{n+1}) \\
&= x + (x^2 \sum_{n=0}^{\infty} F_n x^n) + (x \sum_{n=0}^{\infty} F_n x^n) \qquad \text{because } F_0 = 0 \\
&= x + x^2 f(x) + x f(x)
\end{aligned}
$$

Solution (continued). Reminder: $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_n + F_{n+1}\ \forall n \geq 0$.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} F_n x^n \\
&= F_0 + F_1 x + x^2 \sum_{n=0}^{\infty} F_{n+2} x^n \\
&= x + x^2 \sum_{n=0}^{\infty} (F_n + F_{n+1}) x^n \\
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\end{aligned}$$

So, we got the equation $f(x) = x + x^2 f(x) + x f(x)$, which yields

$$f(x) = -\frac{x}{x^2 + x - 1}.$$

Solution (continued).

$$
\begin{aligned}
f(x) &= -\frac{x}{x^2+x-1} \\
&= -\frac{x}{(x-\frac{-1-\sqrt{5}}{2})(x-\frac{-1+\sqrt{5}}{2})} && \text{via quad. eq.} \\
&= -\frac{\frac{1+\sqrt{5}}{2\sqrt{5}}}{x-\frac{-1-\sqrt{5}}{2}} - \frac{\frac{-1+\sqrt{5}}{2\sqrt{5}}}{x-\frac{-1+\sqrt{5}}{2}} && \text{via partial fractions} \\
&= -\frac{1}{\sqrt{5}}\left(\frac{1}{1-x\frac{1-\sqrt{5}}{2}} - \frac{1}{1-x\frac{1+\sqrt{5}}{2}}\right) && \text{via algebra} \\
&= \frac{1}{\sqrt{5}}\left((-\sum_{n=0}^{\infty}(\frac{1-\sqrt{5}}{2})^n x^n) + (\sum_{n=0}^{\infty}(\frac{1+\sqrt{5}}{2})^n x^n)\right) && \text{via Maclaurin expansion} \\
&= \sum_{n=0}^{\infty} \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}} x^n
\end{aligned}
$$

Solution (continued). So:

$$f(x) = \sum_{n=0}^{\infty} F_n x^n$$

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So, we get:

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for all integers $n \geq 0$.

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So, we get:

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for all integers $n \geq 0$.

We can verify that this works by induction (see the Lecture Notes).

- We defined the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ recursively as follows:
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  for all integers $n \geq 0$.
- The *golden ratio* is the number

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

- We defined the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ recursively as follows:
  - $F_0 = 0$, $F_1 = 1$;
  - $F_{n+2} = F_n + F_{n+1}$ for all integers $n \geq 0$.
- We used generating functions to obtain the following closed formula for the general term of the Fibonacci sequence:

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}.$$

  for all integers $n \geq 0$.
- The *golden ratio* is the number

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

- We have (check this!) that:

$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}.$$

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Example

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence defined recursively as follows:

- $a_0 = 1$;
- $a_{n+1} = 7a_n + 6^{n+1}$ for all integers $n \geq 0$.

Find a closed formula for a_n.

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Solution. We consider the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$ for the sequence $\{a_n\}_{n=0}^{\infty}$. We manipulate $a(x)$ as follows.

Solution (continued). Reminder: $a_0 = 1$, $a_{n+1} = 7a_n + 6^{n+1}$ $\forall n \geq 0$.

$$
\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=0}^{\infty} a_{n+1} x^{n+1} \\
&= 1 + \sum_{n=0}^{\infty} (7a_n + 6^{n+1}) x^{n+1} \\
&= 1 + 7x\Big(\sum_{n=0}^{\infty} a_n x^n\Big) + \Big(\sum_{n=1}^{\infty} 6^n x^n\Big) \\
&= 7x\Big(\sum_{n=0}^{\infty} a_n x^n\Big) + \Big(\sum_{n=0}^{\infty} 6^n x^n\Big) \\
&= 7xa(x) + \tfrac{1}{1-6x}.
\end{aligned}
$$

Solution(continued). So, we got:

$$a(x) = 7xa(x) + \frac{1}{1-6x},$$

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We now compute

$$\begin{aligned} a(x) &= \tfrac{1}{(7x-1)(6x-1)} \\ &= \tfrac{7}{1-7x} - \tfrac{6}{1-6x} && \text{via partial fractions} \\ &= (7\sum_{n=0}^{\infty} 7^n x^n) - (6\sum_{n=0}^{\infty} 6^n x^n) \\ &= \sum_{n=0}^{\infty} (7^{n+1} - 6^{n+1}) x^n. \end{aligned}$$

Solution (continued). So, we have

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We can check by induction that this is correct.