NDMI011: Combinatorics and Graph Theory 1

Lecture #1

Asymptotic notation. Estimates of factorials and binomial coefficients

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September 29, 2021

- Asymptotic comparison of functions

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- Let us try to formalize this.

Definition

Given functions $f, g : \mathbb{N} \to \mathbb{R}$ (in practice, we generally assume f, g are positive-valued), notation

$$f(n) = O(g(n))$$

means that there exist constants $n_0 \in \mathbb{N}$ and $C \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq n_0$, then

$$|f(n)| \leq Cg(n).$$

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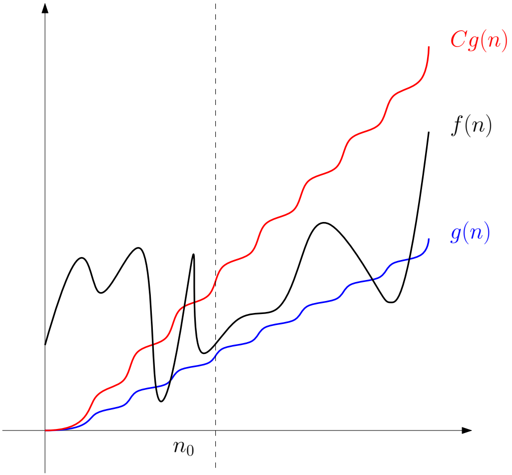

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- Examples:
  1. $10n^2 + 5 = O(n^2)$;
  2. $\ln n + 5 = O(n)$;
  3. $n\sqrt{n} = O(n^2)$.

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$f(n) = o(g(n))$	$\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = 0$
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1. $12n^2 + n = O(n^2)$
2. $n = o(n^2)$
3. $\frac{1}{12}n^3 = \Omega(n^2)$
4. $\frac{1}{12}n^2 = \Theta(n^2)$
5. $5n^2 + n \sim 5n^2 + \log n$

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- $f(n) = \Theta(g(n))$ is **not** the same as $f(n) \sim g(n)$.
- For instance, $2n^2 = \Theta(n^2)$, but $2n^2 \not\sim n^2$.

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- For example, $n^4 + n \ln n = n^4 + O(n^2)$ because $n \ln n = O(n^2)$.
- We use similar notation for the symbols o, Ω, and Θ from the table above.

Notation	Meaning
$O(1)$	constant (or bounded above by a constant)
$O(\log n)$	logarithmic (or sublogarithmic)
$O(n)$	linear (or sublinear)
$O(n^2)$	quadratic (or subquadratic)
$O(n^3)$	cubic (or subcubic)
$n^{O(1)}$	polynomial (or subpolynomial)
$2^{O(n)}$	exponential (or subexponential)

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For a positive integer n, we define $n!$ (read "n factorial") to be

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- $n!$ is the number of ways that n distinct objects can be arranged in a sequence.
  - there are n choices for the first term of the sequence, $n-1$ choices for the second, $n-2$ for the third, etc.
- For instance, there are $3! = 6$ ways to arrange the elements of $\{a, b, c\}$ in a sequence, namely:

1. a, b, c
2. a, c, b
3. b, a, c
4. b, c, a
5. c, a, b
6. c, b, a

- For small values of n, computing $n!$ is quite straightforward:
  - $0! = 1$
  - $1! = 1$
  - $2! = 2 \cdot 1 = 2$
  - $3! = 3 \cdot 2 \cdot 1 = 6$
  - $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
  - $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
  - $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
  - $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$
  - $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$
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- However, $n!$ is a very fast growing function, and so computing it for even moderately large n is impractical.
- How about some estimates (upper and lower bounds)?

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- Our goal is to obtain two better estimates for $n!$, as follows:

  (i) $n^{n/2} \leq n! \leq (\frac{n+1}{2})^n$ for all non-negative integers n;

  (ii) $e(\frac{n}{e})^n \leq n! \leq en(\frac{n}{e})^n$ for all positive integers n.

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- In fact, we'll only prove the upper bounds. (See the Lecture Notes for the lower bounds.)

- We first prove the upper bound from (i).

Theorem 2.1

For all non-negative integers n, the following holds:

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- We'll need the inequality of arithmetic and geometric means.

Inequality of arithmetic and geometric means

All non-negative real numbers x and y satisfy

$$\sqrt{xy} \quad \leq \quad \tfrac{x+y}{2}.$$

Proof: Lecture Notes.

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Proof of the upper bound.

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Proof of the upper bound. The statement is obviously true for $n = 0$ and $n = 1$. For an integer $n \geq 2$:

$$\begin{aligned}
n! &= \sqrt{\Big(n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1\Big)\Big(1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n\Big)} \\
&= \Big(\sqrt{n \cdot 1}\Big)\Big(\sqrt{(n-1) \cdot 2}\Big) \ldots \Big(\sqrt{2 \cdot (n-1)}\Big)\Big(\sqrt{1 \cdot n}\Big) \\
&\overset{\text{GM}\leq\text{AM}}{\leq} \tfrac{n+1}{2} \cdot \tfrac{(n-1)+2}{2} \cdot \cdots \cdot \tfrac{2+(n-1)}{2} \cdot \tfrac{1+n}{2} \\
&= (\tfrac{n+1}{2})^n.
\end{aligned}$$

- We now prove the upper bound from (ii).

Theorem 2.3

For all positive integers n, the following holds:

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- We will use the following inequality (which can be proven using calculus).

Proposition 2.2

For all real numbers x, we have $1 + x \leq e^x$.

Proof: Lecture Notes.

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$$\begin{aligned} (n+1)! &= (n+1) \cdot n! \\ &\leq (n+1) \cdot en(\tfrac{n}{e})^n && \text{by ind. hyp.} \\ &= \left(e(n+1)(\tfrac{n+1}{e})^{n+1}\right) \cdot (\tfrac{n}{n+1})^{n+1} e. \end{aligned}$$

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\end{aligned}$$

It remains to show that $(\frac{n}{n+1})^{n+1} e \leq 1$.

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Proof of the upper bound (continued). WTS $(\frac{n}{n+1})^{n+1}e \leq 1$.

$$\begin{aligned}
(\tfrac{n}{n+1})^{n+1}e &= (1 - \tfrac{1}{n+1})^{n+1}e \\
&\leq (e^{-\frac{1}{n+1}})^{n+1}e && \text{by Proposition 2.2 } (1 + x \leq e^x \ \forall x \in \mathbb{R}) \text{ for } x = -\tfrac{1}{n+1} \\
&= 1.
\end{aligned}$$

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Stirling's formula

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

Proof omitted.

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$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k\cdot(k-1)\cdot\cdots\cdot 1} = \prod_{i=0}^{k-1} \frac{n-i}{k-i}.$$

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- Remark: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\binom{n}{k} = \binom{n}{n-k}$.

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- Remark: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\binom{n}{k} = \binom{n}{n-k}$.
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- Remark: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\binom{n}{k} = \binom{n}{n-k}$.
- $\binom{n}{k}$ is the number of k-element subsets of an n-element set.
- For example, the number of 3-element subsets of the 5-element set $\{a, b, c, d, e\}$ is $\binom{5}{3} = 10$:

1. $\{a, b, c\}$
2. $\{a, b, d\}$
3. $\{a, b, e\}$
4. $\{a, c, d\}$
5. $\{a, c, e\}$
6. $\{a, d, e\}$
7. $\{b, c, d\}$
8. $\{b, c, e\}$
9. $\{b, d, e\}$
10. $\{c, d, e\}$

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Binomial theorem

For all integers $n \geq 0$, and all real numbers x and y, the following holds:

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \cdots + \binom{n}{n-1} x^{n-1} y + \binom{n}{n} x^n.\end{aligned}$$

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- As in the case of factorials, binomial coefficients are easy to compute for small values of n and k. However, even for moderately large n, k, computing $\binom{n}{k}$ becomes impractical.
- So, as in the case of factorials, we would like to obtain some useful estimates (convenient upper and lower bounds) for binomial coefficients.

Theorem 3.1

For all integers n and k s.t. $n \geq k \geq 1$, the following holds:

$$\left(\frac{n}{k}\right)^k \quad \leq \quad \binom{n}{k} \quad \leq \quad \left(\frac{en}{k}\right)^k.$$

Theorem 3.1

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- Theorem 3.1 follows from the two propositions below.

Proposition 3.2

For all integers n and k s.t. $n \geq k \geq 1$, we have that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k}$$

Proposition 3.3

For all integers n and k s.t. $n \geq k \geq 1$, we have that:

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{en}{k}\right)^k.$$

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Proof.

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Proof. Fix integers n, k s.t. $n \geq k \geq 1$. We observe that $\forall i \in \{0, \ldots, k-1\}$, we have that $\frac{n-i}{k-i} \geq \frac{n}{k}$,

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For all integers n and k s.t. $n \geq k \geq 1$, we have that

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Proof. Fix integers n, k s.t. $n \geq k \geq 1$. We observe that $\forall i \in \{0, \ldots, k-1\}$, we have that $\frac{n-i}{k-i} \geq \frac{n}{k}$, and so

$$\binom{n}{k} \;=\; \prod_{i=0}^{k-1} \tfrac{n-i}{k-i} \;\geq\; \prod_{i=0}^{k-1} \tfrac{n}{k} \;=\; \left(\tfrac{n}{k}\right)^k.$$

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Proof. ***Claim.*** *For all real numbers x s.t. $0 < x \leq 1$:*

$$\sum_{i=0}^{k} \binom{n}{i} \quad \leq \quad \frac{(1+x)^n}{x^k}.$$

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Proof of the Claim. By the Binomial theorem:

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \geq \sum_{i=0}^{k} \binom{n}{i} x^i.$$

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Dividing by x^k, we then obtain

$$\frac{(1+x)^n}{x^k} \quad \geq \quad \sum_{i=0}^{k} \binom{n}{i} \frac{1}{x^{k-i}} \quad \overset{0<x\leq 1}{\geq} \quad \sum_{i=0}^{k} \binom{n}{i}$$

This proves the Claim.

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We now compute apply the Claim to $x := \frac{k}{n}$, and we obtain

$$\begin{aligned}
\sum_{i=0}^{k} \binom{n}{i} &\leq (1+\tfrac{k}{n})^n(\tfrac{n}{k})^k && \text{by the Claim for } x = \tfrac{k}{n} \\
&\leq (e^{k/n})^n(\tfrac{n}{k})^k && \text{by Proposition 2.2 for } x = \tfrac{k}{n} \\
& && (1 + x \leq e^x \ \forall x \in \mathbb{R}) \\
&= (\tfrac{en}{k})^k.
\end{aligned}$$

- So, we have proven Theorem 3.1.

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- Which one is the largest?

- For all integers n and k s.t. $n \geq k \geq 1$, we have that

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$$\binom{n}{0} \quad < \quad \binom{n}{1} \quad < \quad \cdots \quad < \quad \binom{n}{n/2} \quad > \quad \cdots \quad > \quad \binom{n}{n-1} \quad > \quad \binom{n}{n},$$

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- In particular, $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$ is maximum among the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.
- Let's find good bounds for $\binom{2m}{m}$.

Theorem 3.4

For all integers $m \geq 1$, we have that

$$\frac{2^{2m}}{2\sqrt{m}} \quad \leq \quad \binom{2m}{m} \quad \leq \quad \frac{2^{2m}}{\sqrt{2m}}$$

Proof.

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Proof. Fix an integer $m \geq 1$, and let

$$P \quad = \quad \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)}.$$

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Then

$$\begin{aligned} P &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)} \\ &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)} \cdot \frac{2 \cdot 4 \cdot \cdots \cdot (2m)}{2 \cdot 4 \cdot \cdots \cdot (2m)} \\ &= \frac{(2m)!}{2^{2m}(m!)^2} \\ &= \frac{1}{2^{2m}} \binom{2m}{m}. \end{aligned}$$

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Proof (continued). Reminder: $P = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)} = \frac{1}{2^{2m}}\binom{2m}{m}$.

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We prove the upper bound for P, as follows:

$$\begin{aligned} 1 &\geq (1-\tfrac{1}{2^2})(1-\tfrac{1}{4^2})\dots(1-\tfrac{1}{(2m)^2}) \\ &= \frac{2^2-1}{2^2}\cdot\frac{4^2-1}{4^2}\cdot\dots\cdot\frac{(2m)^2-1}{(2m)^2} \\ &= \frac{1\cdot 3}{2^2}\cdot\frac{3\cdot 5}{4^2}\cdot\dots\cdot\frac{(2m-1)(2m+1)}{(2m)^2} \\ &= (2m+1)P^2, \end{aligned}$$

which implies $P \leq \frac{1}{\sqrt{2m+1}} \leq \frac{1}{\sqrt{2m}}$.

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which implies $P \leq \frac{1}{\sqrt{2m+1}} \leq \frac{1}{\sqrt{2m}}$. Lower bound: Lecture Notes.

Stirling's formula

$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$

- Finally, we note that using Stirling's formula (which we stated without proof), we can obtain an even better approximation of $\binom{2m}{m}$, as follows:

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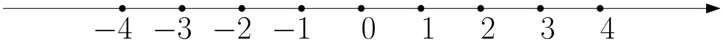
- So, for very large values of m, the function $g(m) = \frac{2^{2m}}{\sqrt{\pi m}}$ is a good approximation of $\binom{2m}{m}$.

- An application: random walks



- We begin our walk at the origin (i.e. 0).

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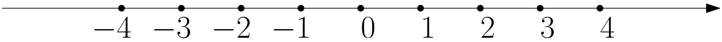

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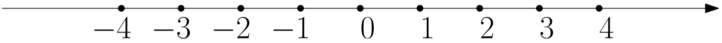

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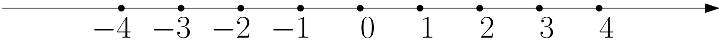

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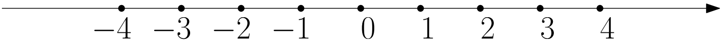
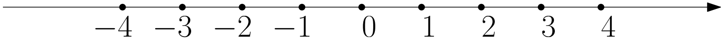
- At each step we move at random either one step to the left (-1) or one step to the right ($+1$), and we continue forever.
- We would like to estimate the number of times that we return to the origin in such a walk.
- Obviously, we can only return to the origin after an even number of steps: the number of time we move to the left should be the same as the number of times we move to the right.



- There are 2^{2m} random walks of length $2m$, and exactly $\binom{2m}{m}$ of those walks end at the origin.
  - Indeed, we must go left exactly m times, and right exactly m times.
  - Out of $2m$ moves, we have $\binom{2m}{m}$ ways of selecting the m leftward moves (the other m moves are rightward).



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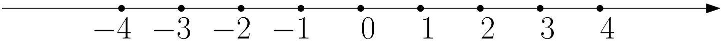

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- So, in an infinite random walk, the expected number of returns to the origin is

$$\sum_{m=1}^{\infty} \frac{\binom{2m}{m}}{2^{2m}}.$$

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- So, the expected number of returns to the origin is

$$\sum_{m=1}^{\infty} \frac{\binom{2m}{m}}{2^{2m}} \geq \sum_{m=1}^{\infty} \frac{1}{2\sqrt{m}} \geq \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} = \infty,$$

where we used the fact that the harmonic series $\sum_{m=1}^{\infty} \frac{1}{m}$ diverges to infinity.