NDMI011: Combinatorics and Graph Theory 1
HW#9

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due Wednesday, January 5, 2022 before midnight (Prague time)

Remark: Please e-mail me (ipenev@iuuk.mff.cuni.cz) your HW as a **PDF attachment** (no other format will be accepted).

Definition. *A* Δ-system *is a collection* $\mathcal{M}$ *of sets, such that any two elements of* $\mathcal{M}$ *have the same intersection.*[1]

Problem 1 (100 points)**.** *Let* $\mathcal{A}$ *be an infinite set of sets, all of which have the same finite cardinality.*[2]

(a) [50 points] Prove that there exists an infinite set $\mathcal{B} \subseteq \mathcal{A}$ *such that any two elements of* $\mathcal{B}$ *have an intersection of the same size.*[3]

Hint: *Ramsey's theorem (infinite version).*

(b) [50 points] Using part (a), prove that $\mathcal{A}$ *contains an infinite* Δ*-system as a subset.*[4]

[1] In other words, a Δ-*system* is a collection $\mathcal{M}$ of sets such that there exists a set K that has the property that all distinct $M_1, M_2 \in \mathcal{M}$ satisfy $M_1 \cap M_2 = K$.

[2] That is: $\mathcal{A}$ is an infinite set, and there exists a non-negative integer k such that every element of $\mathcal{A}$ is a set of size k.

[3] That is: prove that there exists an infinite set $\mathcal{B} \subseteq \mathcal{A}$ and a non-negative integer ℓ such that all distinct $X_1, X_2 \in \mathcal{B}$ satisfy $|X_1 \cap X_2| = \ell$.

[4] That is: prove that there exists an infinite set $\mathcal{M} \subseteq \mathcal{A}$ such that $\mathcal{M}$ is a Δ-system.