NDMI011: Combinatorics and Graph Theory 1

Tutorial #1

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October 6, 2020

- Review of Lecture#1 (from last week).

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Definition

For a positive integer n, we define $n!$ (read "n factorial") to be

$$n! \quad := \quad n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1.$$

Furthermore, as a convention, we set $0! = 1$.

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We proved the following two estimates of the factorial function:

(i) $n^{n/2} \leq n! \leq (\frac{n+1}{2})^n$ for all non-negative integers n;

(ii) $e(\frac{n}{e})^n \leq n! \leq en(\frac{n}{e})^n$ for all positive integers n.

Definition

For integers n and k such that $n \geq k \geq 0$, we define $\binom{n}{k}$, read "n choose k," as follows:

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k\cdot(k-1)\cdot\cdots\cdot 1} = \prod_{i=0}^{k-1} \frac{n-i}{k-i}.$$

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Theorem 2.1

For all integers n and k such that $n \geq k \geq 1$, the following holds:

$$\left(\frac{n}{k}\right)^k \;\leq\; \binom{n}{k} \;\leq\; \left(\frac{en}{k}\right)^k.$$

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$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Theorem 2.4

For all integers $m \geq 1$, we have that

$$\frac{2^{2m}}{2\sqrt{m}} \leq \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{2m}}$$

- We stated Stirling's formula without proof.

Stirling's formula

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n}{n!} = 1.$$

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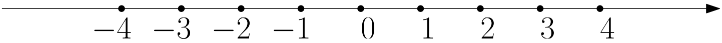
- Stirling's formula readily implies the following:

$$\lim_{m\to\infty} \left(\left(\frac{2^{2m}}{\sqrt{\pi m}}\right) \Big/ \binom{2m}{m} \right) = 1.$$

- An application: random walks



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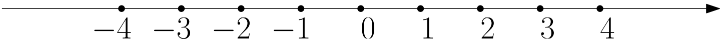

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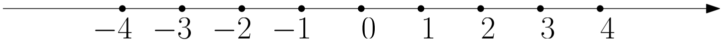

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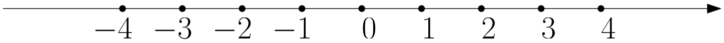

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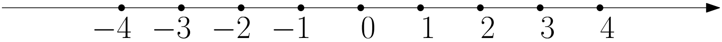
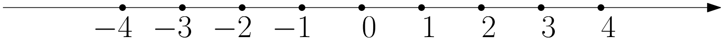
- We would like to estimate the number of times that we return to the origin in such a walk.
- Obviously, we can only return to the origin after an even number of steps: the number of time we move to the left should be the same as the number of times we move to the right.



- There are 2^{2m} random walks of length $2m$, and exactly $\binom{2m}{m}$ of those walks end at the origin.
  - Indeed, we must go left exactly m times, and right exactly m times.
  - Out of $2m$ moves, we have $\binom{2m}{m}$ ways of selecting the m leftward moves (the other m moves are rightward).



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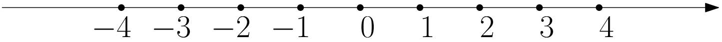

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- So, in an infinite random walk, the expected number of returns to the origin is
$$\sum_{m=1}^{\infty} \frac{\binom{2m}{m}}{2^{2m}}.$$

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- By Theorem 2.4, for all integers $m \geq 1$, we have

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- So, the expected number of returns to the origin is

$$\sum_{m=1}^{\infty} \frac{\binom{2m}{m}}{2^{2m}} \geq \sum_{m=1}^{\infty} \frac{1}{2\sqrt{m}} \geq \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} = \infty,$$

where we used the fact that the harmonic series $\sum_{m=1}^{\infty} \frac{1}{m}$ diverges to infinity.

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Definition

Given functions $f, g : \mathbb{N} \to \mathbb{R}$ (in practice, we generally assume f, g are positive-valued), notation

$$f(n) = O(g(n))$$

means that there exists an integer $n_0 \in \mathbb{N}$ and real number C such that for all $n \in \mathbb{N}$, if $n \geq n_0$, then

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- Examples:
  - (i) $10n^2 + 5 = O(n^2)$;
  - (ii) $\ln n + 5 = O(n)$;
  - (iii) $n\sqrt{n} = O(n^2)$.

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$f(n) = o(g(n))$	$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$
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- $f(n) = \Theta(g(n))$ is **not** the same as $f(n) \sim g(n)$.
- For instance, $2n^2 = \Theta(n^2)$, but $2n^2 \not\sim n^2$.

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- So, Stirling's formula states that

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- and it implies that

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- For example, $n^4 + n \ln n = n^4 + O(n^2)$ because $n \ln n = O(n^2)$.
- We use similar notation for the symbols o, Ω, and Θ from the table above.

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- Determine which of the following are correct.
  (a) $n^2 = O(n^2 \ln n)$
  (b) $n^2 = o(n^2 \ln n)$
  (c) $n^2 + 5n \ln n = n^2(1 + o(1)) \sim n^2$
  (d) $n^2 + 5n \ln n = n^2 + O(n)$
  (e) $n! \sim (\frac{n+1}{2})^n$
  (f) $\ln(n!) = \Omega(n \ln n)$