NDMI011: Combinatorics and Graph Theory 1

Lecture #14

Linear codes

Irena Penev

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- a bit of Linear Algebra;

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- For vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{F}^n$, we define $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$, where the summation and multiplication denote the operations from the field $\mathbb{F}$.
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  - So, $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{F}$.
  - If $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are said to be *orthogonal*.

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- With these adjustments, all familiar theorems of Linear Algebra still hold, but with rows and columns reversed.

Definition

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Theorem 1.1

Let $\mathbb{F}$ be a field, and let C be a subspace of $\mathbb{F}^n$. Then
$\dim C + \dim C^\perp = n$.

Proof (outline). This essentially follows from the Rank-nullity theorem. (Details: Lecture Notes.)

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Proposition 1.2

Let $\mathbb{F}$ be a field, and let C be a subspace of $\mathbb{F}^n$. Then $(C^\perp)^\perp = C$.

Proof. Lecture Notes.

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Since $k = \log_q |C|$ (by definition), it follows that $\ell = k$, which is what we needed to show.

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  - Indeed, if $\mathbf{x}H = \mathbf{0}$, then $\mathbf{x} \in C$, and otherwise, $\mathbf{x} \notin C$.
- Note that, given a generator matrix for C, one can easily compute a parity check matrix for C, and vice versa.

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Since C is a linear code, we know that $\mathbf{0} \in C$, and so (since $\mathbf{x}$ and $\mathbf{0}$ are distinct codewords in C) we have that $d(\mathbf{x}, \mathbf{0}) \geq d$. But obviously, $d(\mathbf{x}, \mathbf{0}) = \mathrm{wt}(\mathbf{x})$, and it follows that $\mathrm{wt}(\mathbf{x}) \geq d$.

WTS $\mathrm{wt}(\mathbf{x}) \leq d$. Fix distinct $\mathbf{y}, \mathbf{z} \in C$ such that $d(\mathbf{y}, \mathbf{z}) = d$. Since C is a vector space, we know that $\mathbf{y} - \mathbf{z} \in C$, and so by the choice of $\mathbf{x}$, we have that $\mathrm{wt}(\mathbf{x}) \leq \mathrm{wt}(\mathbf{y} - \mathbf{z})$.

Definition

Given a vector $\mathbf{x} \in \mathbb{F}_q^n$, the *Hamming weight* of $\mathbf{x}$, denoted by $\mathrm{wt}(\mathbf{x})$, is the number of non-zero coordinates in $\mathbf{x}$.

Proposition 2.2

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    - For the sake of simplicity, though, we consider only binary Hamming codes, i.e. those over the field $\mathbb{F}_2$.
- We do this by constructing its parity check matrix H; then the code in question will simply be the subspace $C = \{\mathbf{x} \in \mathbb{F}_2^n \mid \mathbf{x}H = \mathbf{0}\}$.

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- But $C = \text{Ker}(H)$, and so $\dim C = k$.
    - So, k in $[n, k, d]_2$ is correct.

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- We can correct the error by altering the entry (from 1 to 0, or vice versa) in that one coordinate of $\mathbf{w}$.