NDMI011: Combinatorics and Graph Theory 1

Lecture #12

Bounding the number of edges in graphs without certain subgraphs

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Definition

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Let n be a positive integer. Then

(a) any triangle-free graph on n vertices has at most $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$ edges;

(b) there exists a triangle-free graph on n vertices that has precisely $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$ edges.

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Proof (outline). For (b), we observe that the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is triangle-free and has precisely n vertices and $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ edges.

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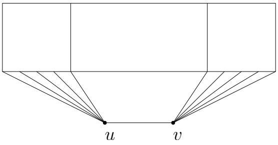
Proof (outline, continued). WMA G has at least one edge (say, uv), for otherwise we are done. No vertex in $V(G) \setminus \{u, v\}$ is adjacent to both u and v (otherwise, we'd get a triangle).



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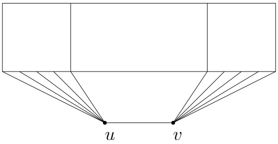
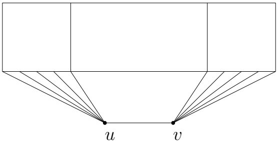
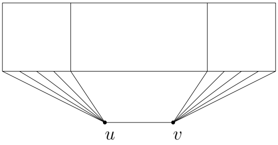


By the induction hypothesis, $|E(G \setminus \{u, v\})| \leq \lfloor \frac{(n-2)^2}{4} \rfloor$.



Proof (outline, continued). Reminder: $|E(G \setminus \{u, v\})| \leq \lfloor \frac{(n-2)^2}{4} \rfloor$.



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Since the edges of G are precisely the edges of $G \setminus \{u,v\}$, plus the edges between $\{u,v\}$ and $V(G) \setminus \{u,v\}$, plus the edge uv, we see that

$$\begin{aligned} |E(G)| &\leq \lfloor \tfrac{(n-2)^2}{4} \rfloor + (n-2) + 1 \\ &= \lfloor \tfrac{n^2-4n+4}{4} \rfloor + n - 1 \\ &= \lfloor \tfrac{n^2}{4} \rfloor, \end{aligned}$$

which is what we needed to show.

The Cauchy-Schwarz inequality

All real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ satisfy

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \quad \leq \quad \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big).$$

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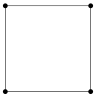
Let $n \in \mathbb{N}$. Any graph on n vertices that does not contain C_4 as a subgraph has at most $\frac{1}{2}(n + n^{3/2})$ edges.



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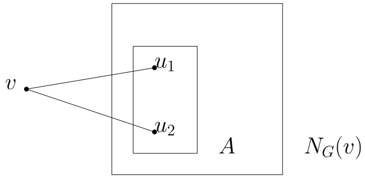
Let $M = \{(v, A) \mid v \in V(G), A \in \binom{N_G(v)}{2}\}$.



Now, we will count the number of elements of M in two ways.

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First, for each $v \in V(G)$, there are precisely $\binom{d_G(v)}{2}$ choices of A such that $(v, A) \in M$. So, $|M| = \sum_{v \in V(G)} \binom{d_G(v)}{2} = \sum_{i=1}^{n} \binom{d_i}{2}$.

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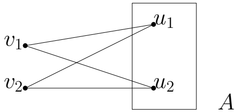
We now bound $|M|$ above, as follows. Since G contains no C_4 as a subgraph, we see that no two distinct elements of M have the same second coordinate. So, $|M| \leq \binom{n}{2}$.



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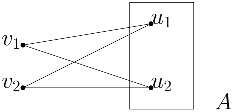

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Obviously, $\binom{n}{2} \leq \frac{n^2}{2}$, and since $d_1, \dots, d_n \geq 1$, we see that $\binom{d_i}{2} \geq \frac{(d_i-1)^2}{2}$ for all $i \in \{1, \dots, n\}$; consequently,

$$\sum_{i=1}^{n} \frac{(d_i-1)^2}{2} \quad \leq \quad \sum_{i=1}^{n} \binom{d_i}{2} \quad \leq \quad \binom{n}{2} \quad \leq \quad \frac{n^2}{2},$$

and it follows that

$$\sum_{i=1}^{n} (d_i - 1)^2 \quad \leq \quad n^2.$$

- Cauchy-Schwarz: $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$.

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$$
\begin{aligned}
\sum_{i=1}^{n}(d_i - 1) &= \sum_{i=1}^{n}(d_i - 1)\cdot 1 \\
&\leq \sqrt{\sum_{i=1}^{n}(d_i - 1)^2}\sqrt{\sum_{i=1}^{n} 1^2} && \text{by C-S} \leq \\
&= \sqrt{\sum_{i=1}^{n}(d_i - 1)^2}\sqrt{n} \\
&\leq \sqrt{n^2}\sqrt{n} && \text{because } \sum_{i=1}^{n}(d_i - 1)^2 \leq n^2 \\
&= n^{3/2}.
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It now follows that

$$|E(G)| \;=\; \frac{1}{2}\sum_{i=1}^{n} d_i \;=\; \frac{1}{2}\Big(n + \sum_{i=1}^{n}(d_i - 1)\Big) \;\leq\; \frac{1}{2}(n + n^{3/2}),$$

which is what we needed to show.