NDMI011: Combinatorics and Graph Theory 1

Lecture #11

Ramsey theory and Kőnig's infinity lemma

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December 21, 2020

- Reminder: For positive integers k and ℓ, $R(k, \ell)$ the smallest $N \in \mathbb{N}$ such that every graph G on at least N vertices satisfies either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

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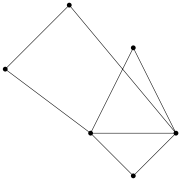
- Numbers $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) are called *Ramsey numbers*, and we proved that they exist in Lecture Notes 10.
- There's another way to think about Ramsey numbers!

- Any graph G corresponds to a complete graph on the same vertex set, and whose edges are colored black or white, with an edge colored black if it was an edge of the graph G, and colored white otherwise.



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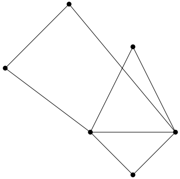

- Now $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) is the smallest $N \in \mathbb{N}$ such that any complete graph on at least N vertices, and whose edges are colored black or white, has either a monochromatic black complete subgraph of size k, or a monochromatic white complete subgraph of size ℓ.

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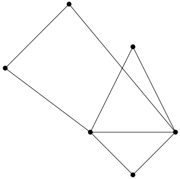

- Now $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) is the smallest $N \in \mathbb{N}$ such that any complete graph on at least N vertices, and whose edges are colored black or white, has either a monochromatic black complete subgraph of size k, or a monochromatic white complete subgraph of size ℓ.
- If instead of black and white, we use colors 1 and 2, then a coloring of the complete graph on vertex set X is simply a function $c : \binom{X}{2} \to [2]$.
  - $\binom{X}{p}$ is the set of all p-element subsets of X.

- So, $R(k,\ell)$ (with $k,\ell \in \mathbb{N}$) is the smallest $N \in \mathbb{N}$ such that for all finite sets X with $|X| \geq N$, and all colorings $c : \binom{X}{2} \to [2]$, either there exists a set $A_1 \in \binom{X}{k}$ such that c assigns color 1 to each set in $\binom{A_1}{2}$, or there exists a set $A_2 \in \binom{X}{\ell}$ such that c assigns color 2 to each set in $\binom{A_2}{2}$.

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- This can be generalized!

Definition

A *hypergraph* is an ordered pair $H = (V(H), E(H))$, where $V(H)$ is some non-empty finite set, and $E(H) \subseteq \mathscr{P}(V(H)) \setminus \{\emptyset\}$. Members of $V(H)$ are called *vertices* and members of $E(H)$ are called *edges* of the hypergraph H.

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For a positive integer p, a hypergraph is *p-uniform* if all its edges have precisely p vertices. A hypergraph is *uniform* if it is p-uniform for some p.

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- So, if H is a p-uniform hypergraph, then $E(H) \subseteq \binom{V(H)}{p}$.
- A graph is simply a 2-uniform hypergraph.

Definition

Given $p, t, k_1, \ldots, k_t \in \mathbb{N}$, the *Ramsey number* $R^p(k_1, \ldots, k_t)$ is the smallest $N \in \mathbb{N}$ (if it exists) such that for all finite sets X with $|X| \geq N$, and all colorings (i.e. functions) $c : \binom{X}{p} \to [t]$,[a] there exist an index $i \in [t]$ and a set $A_i \in \binom{X}{k_i}$ such that c assigns color i to each element of $\binom{A_i}{p}$.

[a] So, c is an assignment of colors to the edges of the "complete" p-uniform hypergraph on vertex set X.

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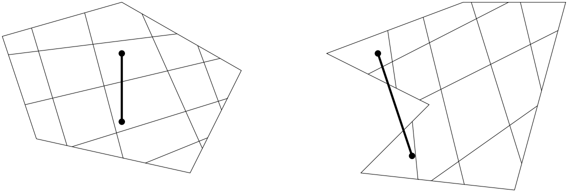
- The other one uses the infinite version of Ramsey's theorem.
- Here, we will present only the second proof.
- But first, let's look at a geometric application!

Definition

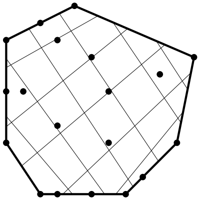
A set X of points in the plane is *convex* if for all distinct $x_1, x_2 \in X$, the line segment between x_1 and x_2 lies in X. The *convex hull* of a non-empty set S of points in the plane is the smallest convex set in the plane that includes S.



convex

non-convex

- If S is a finite set of points in the plane containing at least three non-collinear points, then the convex hull of S is a convex polygon (with its interior), and the vertices of this polygon are all in S.



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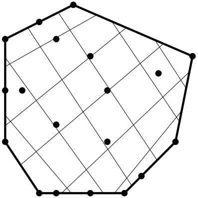

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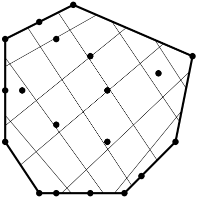

Definition

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- Equivalently, (pairwise distinct) points $x_1, \ldots, x_t$ $(t \geq 3)$ in the plane are in convex position if their convex hull is a convex t-gon whose vertices are precisely $x_1, \ldots, x_t$ (not necessarily in that order).

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Any set of five points in the plane, no three of which are collinear, contains four points in convex position.

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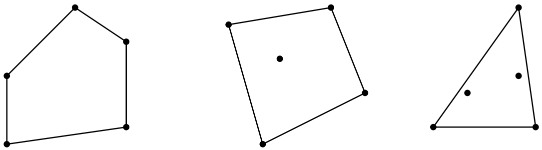

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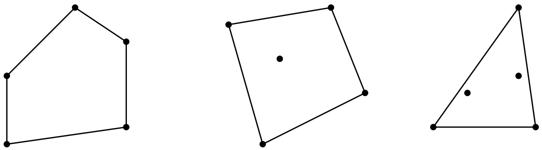

WMA the convex hull is a triangle, for otherwise we are done.

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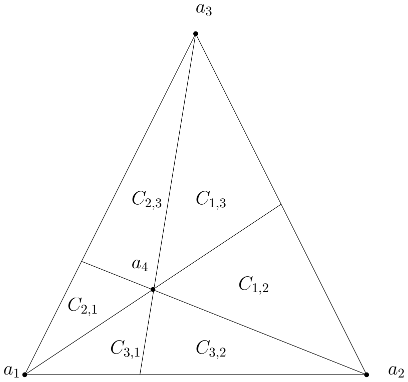

If $a_5 \in C_{i,j}$, then a_i, a_4, a_5, a_j are the vertices of a convex quadrilateral, and we are done.

The Erdős-Szekeres theorem

Let $t \geq 4$ be an integer. Any set of at least $R^4(5, t)$ points in the plane, no three of which are collinear, contains t points in convex position.

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Proof (outline, continued). Suppose that there exists some $A_1 \in \binom{S}{5}$ such that c assigns color 1 to all elements of $\binom{A_1}{4}$.

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Proof (outline, continued). Suppose that there exists some $A_1 \in \binom{S}{5}$ such that c assigns color 1 to all elements of $\binom{A_1}{4}$. Then A_1 is a set of five points in the plane, no three of which are collinear, and no four of which are in convex position. But this contradicts Lemma 1.1.

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Proof (outline, continued).

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Proof (outline, continued). It now follows that there exists some $A_2 \in \binom{S}{t}$ such that c assigns color 2 to all elements of $\binom{A_2}{4}$. Then A_2 is a set of t points in the plane, no three of which are collinear, and any four of which are in convex position.

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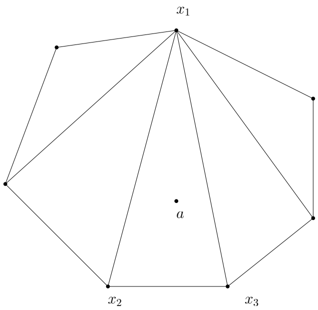

If some point of S is not a vertex of the polygon, then we get four points of A_4 that are not in convex position.

Ramsey's theorem (infinite version)

For all $t, p \in \mathbb{N}$, all infinite sets X, and all colorings $c : \binom{X}{p} \to [t]$, there exists an infinite set $A \subseteq X$ such that $c \restriction \binom{A}{p}$ is constant.

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Proof. We fix $t \in \mathbb{N}$, and we proceed by induction on p.

For $p = 1$, we fix an infinite set X and a coloring $c : \binom{X}{1} \to [t]$. For all $i \in [t]$, we set $C_i = \{x \in X \mid c(\{x\}) = i\}$. Then $(C_1, \dots, C_t)$ is a partition of X, and consequently, at least one of the sets $C_1, \dots, C_t$, say C_i, is infinite. Furthermore, $c \restriction \binom{C_i}{1}$ is constant (indeed, it assigns color i to each element of $\binom{C_i}{1}$). So, the theorem is true for $p = 1$.

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For all $t, p \in \mathbb{N}$, all infinite sets X, and all colorings $c : \binom{X}{p} \to [t]$, there exists an infinite set $A \subseteq X$ such that $c \restriction \binom{A}{p}$ is constant.

Proof (continued). Fix $p \in \mathbb{N}$, and assume the theorem is true for p. We must show that it is true for $p + 1$.

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Proof (continued). Fix $p \in \mathbb{N}$, and assume the theorem is true for p. We must show that it is true for $p+1$. Fix an infinite set X and a coloring $c : \binom{X}{p+1} \to [t]$. Our goal is to recursively construct a sequence $\{X_n\}_{n=1}^{\infty}$ of infinite subsets of X and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X with the following three properties:

- $x_n \in X_n$ for all $n \in \mathbb{N}$;
- $X_{n+1} \subseteq X_n \setminus \{x_n\}$ for all $n \in \mathbb{N}$;
- for all $n \in \mathbb{N}$, c assigns the same color to all sets of the form $\{x_n\} \cup X$, with $X \in \binom{X_{n+1}}{p}$.

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For all $t, p \in \mathbb{N}$, all infinite sets X, and all colorings $c : \binom{X}{p} \to [t]$, there exists an infinite set $A \subseteq X$ such that $c \restriction \binom{A}{p}$ is constant.

Proof (continued). First, we set $X_1 = X$ and we choose $x_1 \in X$ arbitrarily. Now, having constructed $X_1, \ldots, X_n$ and $x_1, \ldots, x_n$, we construct X_{n+1} and x_{n+1} as follows. We define an auxiliary coloring $c_n : \binom{X_n \setminus \{x_n\}}{p} \to [t]$ by setting $c_n(A) = c(A \cup \{x_n\})$ for all $A \in \binom{X_n \setminus \{x_n\}}{p}$. Since $X_n \setminus \{x_n\}$ is infinite, the induction hypothesis guarantees that there exists some infinite set $X_{n+1} \subseteq X_n \setminus \{x_n\}$ such that $c_n \restriction \binom{X_{n+1}}{p}$ is constant. But now by construction, we have that c assigns the same color to all sets of the form $\{x_n\} \cup X$, with $X \in \binom{X_{n+1}}{p}$. Finally, we choose $x_{n+1} \in X_{n+1}$ arbitrarily.

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For all $t, p \in \mathbb{N}$, all infinite sets X, and all colorings $c : \binom{X}{p} \to [t]$, there exists an infinite set $A \subseteq X$ such that $c \restriction \binom{A}{p}$ is constant.

Proof (continued). We have now constructed a sequence $\{X_n\}_{n=1}^{\infty}$ of infinite subsets of X and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X with the following three properties:

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It follows from the construction that for all $n \in \mathbb{N}$, the coloring c assigns the same color to all sets of the form $\{x_n\} \cup \{x_{j_1}, \ldots, x_{j_p}\}$, with $n < j_1 < \cdots < j_p$; let us say this color is *associated* with x_n.

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Proof (continued). Reminder: For all $n \in \mathbb{N}$, the coloring c assigns the same color to all sets of the form $\{x_n\} \cup \{x_{j_1}, \ldots, x_{j_p}\}$, with $n < j_1 < \cdots < j_p$; this color is *associated* with x_n.

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For all $i \in [t]$, we let $A_i = \{x_n \mid n \in \mathbb{N}, i \text{ is associated with } x_n\}$. Then $(A_1, \dots, A_t)$ is a partition of the infinite set $\{x_1, x_2, x_3, \dots\}$, and we deduce that at least one of the sets $A_1, \dots, A_t$, say A_i, is infinite. But now $c \restriction \binom{A_i}{p+1}$ is constant (it assigns i to all elements of $\binom{A_i}{p+1}$). This completes the induction.

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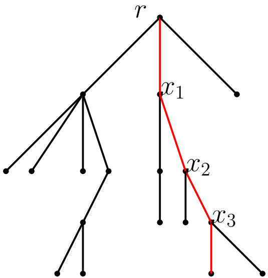

Definition

Given $p, t, k_1, \ldots, k_t \in \mathbb{N}$, the *Ramsey number* $R^p(k_1, \ldots, k_t)$ is the smallest $N \in \mathbb{N}$ (if it exists) such that for all finite sets X with $|X| \geq N$, and all colorings (i.e. functions) $c : \binom{X}{p} \to [t]$,[a] there exist an index $i \in [t]$ and a set $A_i \in \binom{X}{k_i}$ such that c assigns color i to each element of $\binom{A_i}{p}$.

[a]So, c is an assignment of colors to the edges of the "complete" p-uniform hypergraph on vertex set X.

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Proof (continued). Now, let C be the set of all bad colorings, and let T be the graph on the vertex set $C \cup \{r\}$ (where $r \notin C$), with adjacency as follows:

- r is adjacent to all p-bad colorings, and to no other elements of C;
- for all integers $n \geq p$, n-bad colorings are pairwise non-adjacent;
- for all integers $n \geq p$, an n-bad coloring c_n is adjacent to an $(n+1)$-bad coloring c_{n+1} iff c_{n+1} is an extension of c_n;[1]
- for all integers $n_1, n_2 \geq p$ such that $|n_1 - n_2| \geq 2$, no n_1-bad coloring is adjacent to any n_2-bad coloring.

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Now (T, r) is a rooted tree. Furthermore, for each integer $n \geq p$, the number of n-bad colorings is finite, and it follows from the construction of T that the T is locally finite. So, by Kőnig's infinity lemma, there is a ray $r, c_p, c_{p+1}, c_{p+2}, \dots$ in T.

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We now choose any subset $A_k \in \binom{A}{k}$, and we observe that $c \restriction \binom{A_k}{p}$ is constant.

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