NDMI011: Combinatorics and Graph Theory 1

Lecture #11
Ramsey theory and Kőnig's infinity lemma

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1 Ramsey's theorem (hypergraph version)

First, we need some notation. We denote by $\mathbb{N}$ the set of all positive integers.[1] For a positive integer n, we set $[n] = \{1, \dots, n\}$. For a set X and a non-negative integer k, we denote by $\binom{X}{k}$ the set of all subsets of X of size k. In particular, $\binom{X}{2}$ is the set of all subsets of X of size two. Note that this means that if G is a (simple) graph, then $E(G) \subseteq \binom{V(G)}{2}$.

Recall that for positive integers k and ℓ, $R(k, \ell)$ the smallest $N \in \mathbb{N}$ such that every graph G on at least N vertices satisfies either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$. Numbers $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) are called *Ramsey numbers*, and we proved that they exist in Lecture Notes 10.

Here is a slightly different way to think about Ramsey numbers. Clearly, any graph G corresponds to a complete graph on the same vertex set, and whose edges are colored black or white, with an edge colored black if it was an edge of the graph G, and colored white otherwise. With this set-up, it is easy to see that $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) is the smallest $N \in \mathbb{N}$ such that any complete graph on at least N vertices, and whose edges are colored black or white, has either a monochromatic[2] black complete subgraph of size k, or a monochromatic white complete subgraph of size ℓ. Now, let us suppose that instead of colors black and white, we use colors 1 and 2. Then a coloring of the complete graph on vertex set X is simply a function $c : \binom{X}{2} \to [2]$.[3] We now see that $R(k, \ell)$ (with $k, \ell \in \mathbb{N}$) is the smallest $N \in \mathbb{N}$ such that for all finite sets X with $|X| \geq N$, and all colorings $c : \binom{X}{2} \to [2]$, either there exists a set $A_1 \in \binom{X}{k}$ such that c assigns color 1 to each set in $\binom{A_1}{2}$, or there exists a set $A_2 \in \binom{X}{\ell}$ such that c assigns color 2 to each set in $\binom{A_2}{2}$.

[1] In some texts, $\mathbb{N}$ is used to denote the set of all non-negative integers. Here, it is the set of all positive integers.

[2] Here, "monochromatic" simply means that all edges are colored with the same color.

[3] Note that the edge set of the complete graph on vertex set X is precisely the set $\binom{X}{2}$.

This can be generalized!

A *hypergraph* is an ordered pair $H = (V(H), E(H))$, where $V(H)$ is some non-empty finite set,[4] and $E(H) \subseteq \mathscr{P}(V(H)) \setminus \{\emptyset\}$. As in the graph case, members of $V(H)$ are called *vertices* and members of $E(H)$ are called *edges* of the hypergraph H. For a positive integer p, a hypergraph is p-*uniform* if all its edges have precisely p vertices. A hypergraph is *uniform* if it is p-uniform for some p. So, if H is a p-uniform hypergraph, then $E(H) \subseteq \binom{V(H)}{p}$. Note that this means that a graph is simply a 2-uniform hypergraph.

Given $p, t, k_1, \dots, k_t \in \mathbb{N}$, the *Ramsey number* $R^p(k_1, \dots, k_t)$ is the smallest $N \in \mathbb{N}$ (if it exists) such that for all finite sets X with $|X| \geq N$, and all colorings (i.e. functions) $c : \binom{X}{p} \to [t]$,[5] there exist an index $i \in [t]$ and a set $A_i \in \binom{X}{k_i}$ such that c assigns color i to each element of $\binom{A_i}{p}$.[6] If no such N exists, then $R^p(k_1, \dots, k_t)$ is undefined. As the next theorem shows, the Ramsey numbers $R^p(k_1, \dots, k_t)$ are always defined.

Ramsey's theorem (hypergraph version). *For all $p, t, k_1, \dots, k_t \in \mathbb{N}$, the number $R^p(k_1, \dots, k_t)$ exists.*

Proof. We fix $t \in \mathbb{N}$, and we proceed by induction on p.

First, for $p = 1$, we fix $k_1, \dots, k_t \in \mathbb{N}$, and we set $N = (k_1 - 1) + \dots + (k_t - 1) + 1$. Fix any finite set X with $|X| \geq N$, and any coloring $c : \binom{X}{p} \to [t]$. Now, for all $i \in [t]$, set $C_i = \{x \in X \mid c(\{x\}) = i\}$. Then $(C_1, \dots, C_t)$ is a partition of X, and $|X| \geq N = (k_1 - 1) + \dots + (k_t - 1) + 1$. So, by the Pigeonhole Principle, there is some $i \in [t]$ such that $|C_i| \geq k_i$. Now, let A_i be any subset of C_i such that $|A_i| = k_i$; so, $A_i \in \binom{X}{k_i}$. Furthermore, by construction, c assigns color i to each element of $\binom{A_i}{p}$. So, $R^1(k_1, \dots, k_t)$ exists, and we see that the theorem holds for $p = 1$.

Now, fix $p \in \mathbb{N}$, and assume inductively that the Ramsey number $R^p(k_1, \dots, k_t)$ is defined for all $k_1, \dots, k_t \in \mathbb{N}$. We must show that the number $R^{p+1}(k_1, \dots, k_t)$ is defined for all $k_1, \dots, k_t \in \mathbb{N}$.

Fix $k_1, \dots, k_t \in \mathbb{N}$, and assume inductively that the number $R^p(k'_1, \dots, k'_t)$ is defined for all $k'_1, \dots, k'_t \in \mathbb{N}$ such that $k'_1 + \dots + k'_t < k_1 + \dots + k_t$.

To simplify notation, we set $r_i = R^{p+1}(k_1, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_t)$ for all $i \in [t]$ (this is defined by the induction hypothesis for $k_1 + \dots + k_t$). Further, we set $N = R^p(r_1, \dots, r_t) + 1$ (this is defined by the induction hypothesis for p).

Fix a finite set X such that $|X| \geq N$, and fix a function $c : \binom{X}{p} \to [t]$. Set $n = |X|$; we may assume that $X = [n]$.[7] We now define an auxiliary coloring $\widetilde{c} : \binom{[n-1]}{p} \to \{t\}$, as follows: for all $A \in \binom{[n-1]}{p}$, we set $\widetilde{c}(A) = c(A \cup \{n\})$.

[4] Occasionally, $V(H)$ is allowed to be empty.

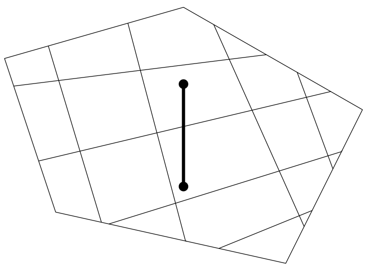
[5] So, c is an assignment of colors to the edges of the "complete" p-uniform hypergraph on vertex set X.

[6] With this set-up, we have that $R(k, \ell) = R^2(k, \ell)$.

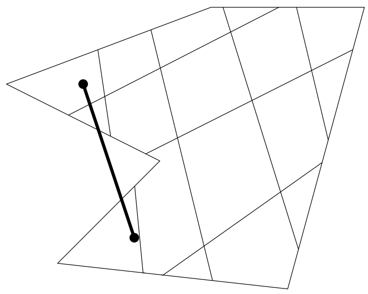
[7] If not, we simply rename the elements of X (via a bijection).

Since $n-1 \geq R^p(r_1, \ldots, r_t)$, we know that there exists some $i \in [t]$ and a set $X_i \in \binom{[n-1]}{r_i}$ such that $\widetilde{c}$ assigns color i to each element of $\binom{X_i}{p}$. Finally, since $|X_i| = r_i = R^{p+1}(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_t)$, we know that there exists some $j \in [t]$ and a set $Y_j \in \binom{X_i}{k'_j}$, where $k'_j = k_j - 1$ if $j = i$ and $k'_j = k_j$ otherwise, such that c assigns color j to each element of $\binom{Y_j}{p+1}$. If $j \neq i$, then we set $A_j = Y_j$, and we observe that $A_j \in \binom{[n]}{k_j}$, and that (by construction) c assigns color j to each element of $\binom{A_j}{p+1}$. Suppose now that $j = i$. Then we set $A_i = Y_i \cup \{n\}$. Once again by construction, we have that $|A_i| = k_i$, and that c assigns color i to each element of $\binom{A_i}{p+1}$.[8] This proves that $R^{p+1}(k_1, \ldots, k_t)$ is defined. $\square$

We now consider a geometric application (see the Erdős-Szekeres theorem below). A set X of points in the plane is *convex* if for all distinct $x_1, x_2 \in X$, the line segment between x_1 and x_2 lies in X.



convex



non-convex

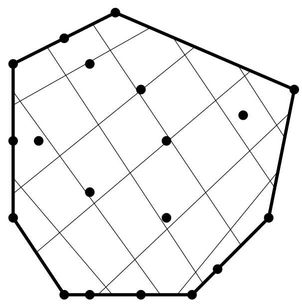
The *convex hull* of a non-empty set S of points in the plane is the smallest convex set in the plane that includes S. If S is a non-empty, finite set of points, then the convex hull of S is either a one-point set, a line interval, or a convex polygon (with its interior).

If S is a finite set of points in the plane containing at least three non-collinear points,[9] then the convex hull of S is a convex polygon (with its interior), and the vertices of this polygon are all in S;[10] see the picture below for an example.

[8] Indeed, fix any $A \in \binom{A_i}{p+1}$. If $n \notin A$, then $A \in \binom{Y_i}{p+1}$, and so $c(A) = i$. On the other hand, if $n \in A$, then $A \setminus \{n\} \in \binom{X_i}{p}$, and we see that $c(A) = \widetilde{c}(A \setminus \{n\}) = i$.

[9] Three or more points are *collinear* if they lie on the same line.

[10] However, not every element of S need be a vertex of the polygon.



Let us say that (pairwise distinct) points $x_1, \ldots, x_t$ ($t \geq 3$) in the plane are in *convex position* if they are the vertices of some convex polygon. Equivalently, (pairwise distinct) points $x_1, \ldots, x_t$ ($t \geq 3$) are in convex position if their convex hull is a convex t-gon whose vertices are precisely $x_1, \ldots, x_t$ (not necessarily in that order).

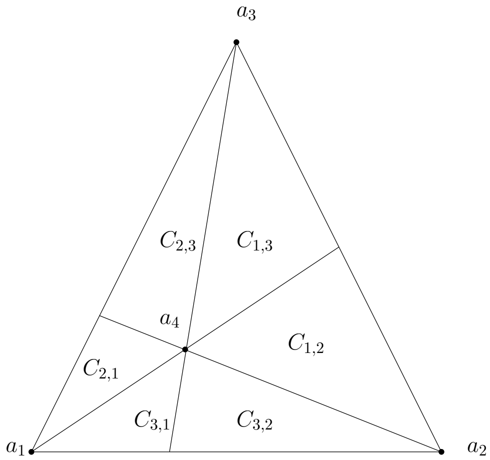
We now need a geometric lemma.

Lemma 1.1. *Any set of five points in the plane, no three of which are collinear, contains four points in convex position.*

Proof. To simplify notation, for non-collinear points x, y, z in the plane, we denote by Δxyz the triangle with vertices x, y, z.

Let $a_1, \ldots, a_5$ be five point in the plane, no three of which are collinear. We now consider the convex hull of these five points. Since no three of these points are collinear, their convex hull is a convex polygon, and each vertex of the polygon is one of $a_1, \ldots, a_5$.[11] If the polygon is a pentagon, then clearly, any four of our five points are in convex position. If the polygon is a quadrilateral, then its vertices (which are some four of $a_1, \ldots, a_5$) are in convex position. So assume that the polygon is a triangle. By symmetry, we may assume that the vertices of this triangle are a_1, a_2, a_3. Since no three points of $a_1, \ldots, a_5$ are collinear, we see that a_4, a_5 both lie in the interior (and not on any edge) of the triangle $\Delta a_1a_2a_3$. Using the fact that a_4 is in the interior of $\Delta a_1a_2a_3$, we construct six regions in the interior of $\Delta a_1a_2a_3$, as in the picture below (the regions $C_{i,j}$ are disjoint from the lines represented in the picture).

[11] However, not all of $a_1, \ldots, a_5$ need be vertices of the polygon.



Since no three of $a_1, \dots, a_5$ are collinear, we see that $a_5 \in C_{1,2} \cup C_{1,3} \cup C_{2,1} \cup C_{2,3} \cup C_{3,1} \cup C_{3,2}$. Now, fix $i, j \in \{1, 2, 3\}$ with $i \neq j$ such that $a_5 \in C_{i,j}$. Then a_i, a_4, a_5, a_j are the vertices of a convex quadrilateral, and we are done. $\square$

The Erdős-Szekeres theorem. *Let $t \geq 4$ be an integer. Any set of at least $R^4(5, t)$ points in the plane, no three of which are collinear, contains t points in convex position.*

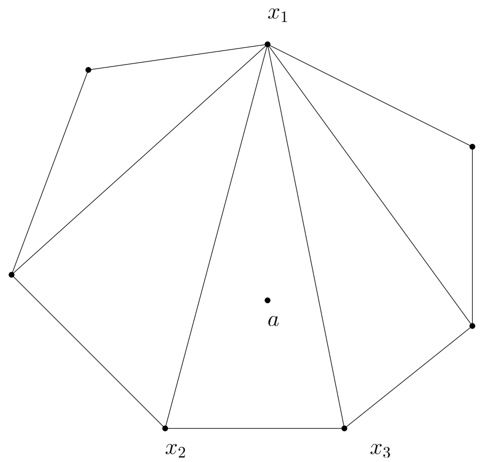
Proof. We consider a set S of at least $R^4(5, t)$ points in the plane, and we assume that no three of these points are collinear. We now consider a coloring $c : \binom{S}{4} \to [2]$ defined as follows: for all $X \in \binom{S}{4}$, $c(X) = 1$ if the four points of X are **not** in convex position, and $c(X) = 2$ if they are in convex position. Since $|S| \geq R^4(5, t)$, we know that either there exists some $A_1 \in \binom{S}{5}$ such that c assigns color 1 to all elements of $\binom{A_1}{4}$, or there exists some $A_2 \in \binom{S}{t}$ such that c assigns color 2 to all elements of $\binom{A_2}{4}$.

Suppose that there exists some $A_1 \in \binom{S}{5}$ such that c assigns color 1 to all elements of $\binom{A_1}{4}$. Then A_1 is a set of five points in the plane, no three of which are collinear, and no four of which are in convex position. But this contradicts Lemma 1.1.

It now follows that there exists some $A_2 \in \binom{S}{t}$ such that c assigns color 2 to all elements of $\binom{A_2}{4}$. Then A_2 is a set of t points in the plane, no three of which are collinear, and any four of which are in convex position. Let us show that the points in A_2 are in fact in convex position. We now consider the convex hull of A_2; this convex hull is a convex polygon, and we let X_2 be the set of vertices of this polygon. Clearly, $X_2 \subseteq A_2$. If $X_2 = A_2$, then we are done. So assume that $X_2 \subsetneqq A_2$. Then all points in $X_2 \setminus A_2$ are in the interior of our polygon.[12] We now choose any $a \in A_2 \setminus X_2$. Clearly, there

[12] Since no three points in A_2 are collinear, no point of $X_2 \setminus A_2$ is on an edge of the polygon.

exist three (pairwise distinct) points $x_1, x_2, x_3 \in X_2$ such that a is in the interior of the triangle $\Delta x_1 x_2 x_3$.[13]



But then a, x_1, x_2, x_3 are not in convex position, contrary to the fact that $\{a, x_1, x_2, x_3\} \in \binom{A_2}{4}$. $\square$

2 Ramsey's theorem (infinite version)

For a function $c : A \to B$ and a set $A' \subseteq A$, we denote by $c \restriction A'$ the restriction of c to A'.[14]

Ramsey's theorem (infinite version). *For all $t, p \in \mathbb{N}$, all infinite sets X, and all colorings $c : \binom{X}{p} \to [t]$, there exists an infinite set $A \subseteq X$ such that $c \restriction \binom{A}{p}$ is constant.*[15]

Proof. We fix $t \in \mathbb{N}$, and we proceed by induction on p.

For $p = 1$, we fix an infinite set X and a coloring $c : \binom{X}{1} \to [t]$. For all $i \in [t]$, we set $C_i = \{x \in X \mid c(\{x\}) = i\}$. Then $(C_1, \dots, C_t)$ is a partition of X, and consequently, at least one of the sets $C_1, \dots, C_t$, say C_i, is infinite. Furthermore, $c \restriction \binom{C_i}{1}$ is constant (indeed, it assigns color i to each element of $\binom{C_i}{1}$). So, the theorem is true for $p = 1$.

Now, fix $p \in \mathbb{N}$, and assume the theorem is true for p.[16] We must show that it is true for $p+1$. Fix an infinite set X and a coloring $c : \binom{X}{p+1} \to [t]$. Our goal is to recursively construct a sequence $\{X_n\}_{n=1}^{\infty}$ of infinite subsets of X and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X with the following three properties:

[13]Once again, we are using the fact that no three of our points are collinear.

[14]So, $c \restriction A'$ is a function from A' to B, and for all $a \in A'$, we have $(c \restriction A')(a) = c(a)$.

[15]This means that c assigns the same color to all p-element subsets of A.

[16]So, we are assuming that for all infinite sets X, and all colorings $c : \binom{X}{p} \to [t]$, there exists an infinite set $A \subseteq X$ such that $c \restriction \binom{A}{p}$ is constant.

- $x_n \in X_n$ for all $n \in \mathbb{N}$;
- $X_{n+1} \subseteq X_n \setminus \{x_n\}$ for all $n \in \mathbb{N}$;
- for all $n \in \mathbb{N}$, c assigns the same color to all sets of the form $\{x_n\} \cup X$, with $X \in \binom{X_{n+1}}{p}$.

First, we set $X_1 = X$ and we choose $x_1 \in X$ arbitrarily. Now, having constructed $X_1, \dots, X_n$ and $x_1, \dots, x_n$, we construct X_{n+1} and x_{n+1} as follows. We define an auxiliary coloring $c_n : \binom{X_n \setminus \{x_n\}}{p} \to [t]$ by setting $c_n(A) = c(A \cup \{x_n\})$ for all $A \in \binom{X_n \setminus \{x_n\}}{p}$.[17] Since $X_n \setminus \{x_n\}$ is infinite, the induction hypothesis guarantees that there exists some infinite set $X_{n+1} \subseteq X_n \setminus \{x_n\}$ such that $c_n \upharpoonright \binom{X_{n+1}}{p}$ is constant. But now by construction, we have that c assigns the same color to all sets of the form $\{x_n\} \cup X$, with $X \in \binom{X_{n+1}}{p}$. Finally, we choose $x_{n+1} \in X_{n+1}$ arbitrarily.

We have now constructed our sequences $\{X_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$. It follows from the construction that for all $n \in \mathbb{N}$, the coloring c assigns the same color to all sets of the form $\{x_n\} \cup \{x_{j_1}, \dots, x_{j_p}\}$, with $n < j_1 < \dots < j_p$; let us say this color is *associated* with x_n. Now, for all $i \in [t]$, we let $A_i = \{x_n \mid n \in \mathbb{N}, i \text{ is associated with } x_n\}$. Then $(A_1, \dots, A_t)$ is a partition of the infinite set $\{x_1, x_2, x_3, \dots\}$, and we deduce that at least one of the sets $A_1, \dots, A_t$, say A_i, is infinite. But now $c \upharpoonright \binom{A_i}{p+1}$ is constant (it assigns i to all elements of $\binom{A_i}{p+1}$). This completes the induction. $\square$

Note that, to form the sequence $\{x_n\}_{n=1}^{\infty}$ in the proof that we just completed, we made infinitely many "arbitrary choices" (indeed, each x_n was chosen arbitrarily from some specified infinite set). So, we implicitly used the "Axiom of Choice," which allows us to make infinitely many arbitrary choices in this way. It is actually possible to avoid the use of the Axiom of Choice in the proof above, but then the proof would be slightly messier,[18] and we omit the details.

3 Kőnig's infinity lemma

An infinite graph (i.e. graph with an infinite vertex set) is *locally finite* if each vertex has finite degree. As in the case of finite graphs, an infinite graph is *connected* if there is a path[19] between any two vertices. An infinite graph is a *forest* if it contains no cycles,[20] and it is a *tree* if it is a connected forest.

[17] Note that if $A \in \binom{X_n \setminus \{x_n\}}{p}$, then $A \cup \{x_n\} \in \binom{X_n}{p+1} \subseteq \binom{X}{p+1}$, and so $c(A \cup \{x_n\})$ is defined.

[18] Essentially, we would start with an injection $f : \mathbb{N} \to X$, and then work with $f[\mathbb{N}]$ instead of X. Then, instead of making an arbitary choice, we could choose the $x_n \in X_n$ whose pre-image (via f) is minimum.

[19] The path is still finite.

[20] Again, cycles are finite.

An *infinite rooted tree* is an ordered pair (T, r) such that T is an infinite tree, and r is some vertex of T, called the *root*.

A *ray* in an infinite graph G is a sequence $x_0, x_1, x_2, x_3, \dots$ of pairwise distinct vertices such that for all integers $n \geq 0$, $x_n x_{n+1}$ is an edge of G.

Kőnig's infinity lemma 1. *Every infinite, locally finite rooted tree (T, r) contains a ray starting at r (i.e. a ray of the form $r, x_1, x_2, \dots$).*

Proof (outline). Since the tree T is infinite, there are infinitely many paths in it with one endpoint r. Since r has only finitely many neighbors, infinitely many of these paths have the second vertex (say, x_1) in common as well. Since x_1 has only finitely many neighbors, among the infinitely many paths starting with r, x_1, infinitely many have the third vertex (say, x_2) in common. We proceed like this, and we obtain an infinite sequence $r, x_1, x_2, x_3, \dots$. But now $r, x_1, x_2, x_3, \dots$ is a ray starting at r. $\square$

We remark that the proof of Kőnig's infinity lemma also uses the Axiom of Choice (because at the n-th step, there may be more than one possible choice for x_n, and if so, we choose arbitrarily).

The infinite version of Ramsey's theorem and Kőnig's infinity lemma together imply the hypergraph version of Ramsey's theorem, as we now show.

Ramsey's theorem (hypergraph version). *For all $p, t, k_1, \dots, k_t \in \mathbb{N}$, the number $R^p(k_1, \dots, k_t)$ exists.*

Proof. Clearly, it suffices to show that for all $p, t, k \in \mathbb{N}$, the Ramsey number $R^p(\underbrace{k, \dots, k}_{t})$ exists.[21] Suppose that for some $p, t, k \in \mathbb{N}$, the number $R^p(\underbrace{k, \dots, k}_{t})$ does not exist. Now, for each integer $n \geq p$, we say that a coloring $c : \binom{[n]}{p} \to [t]$ is *n-bad* if there is no set $A \in \binom{[n]}{k}$ such that $c \restriction \binom{A}{p}$ is constant; a coloring is *bad* if it is n-bad for some integer $n \geq p$. Since $R^p(\underbrace{k, \dots, k}_{t})$ does not exist, we see that for all integers $n \geq p$, there is at least one n-bad coloring.[22]

Now, let C be the set of all bad colorings, and let T be the graph on the vertex set $C \cup \{r\}$ (where $r \notin C$),[23] with adjacency as follows:

- r is adjacent to all p-bad colorings, and to no other elements of C;
- for all integers $n \geq p$, n-bad colorings are pairwise non-adjacent;

[21] Indeed, fix $p, t, k_1, \dots, k_t \in \mathbb{N}$, and set $k = \max\{k_1, \dots, k_t\}$. If $R^p(\underbrace{k, \dots, k}_{t})$ exists, then so does $R^p(k_1, \dots, k_t)$ (details?).

[22] Details?

[23] Here, r is simply an artificially added root, which we need in order to make a rooted tree.

- for all integers $n \geq p$, an n-bad coloring c_n is adjacent to an $(n+1)$-bad coloring c_{n+1} if and only if c_{n+1} is an extension of c_n;[24]
- for all integers $n_1, n_2 \geq p$ such that $|n_1 - n_2| \geq 2$, no n_1-bad coloring is adjacent to any n_2-bad coloring.

Now (T, r) is a rooted tree. Furthermore, for each integer $n \geq p$, the number of n-bad colorings is finite, and it follows from the construction of T that the T is locally finite. So, by Kőnig's infinity lemma, there is a ray $r, c_p, c_{p+1}, c_{p+2}, \ldots$ in T. Set $c = \bigcup_{n=p}^{\infty} c_n$; then $c : \binom{\mathbb{N}}{p} \to [t]$,[25] and so by the infinite version of Ramsey's theorem, there is an infinite set A such that $c \restriction \binom{A}{p}$ is constant. We now choose any subset $A_k \in \binom{A}{k}$, and we observe that $c \restriction \binom{A_k}{p}$ is constant. Now, A_k is a finite subset of $\mathbb{N}$, and consequently, there exists some $n \in \mathbb{N}$ such that $A_k \subseteq [n]$; we may assume that $n \geq p$.[26] Now $A_k \in \binom{[n]}{k}$, and $c_n \restriction \binom{A_k}{p} = c \restriction \binom{A_k}{p}$ is constant, contrary to the fact that c_n is bad. $\square$

[24] This means that $c_{n+1} \restriction \binom{[n]}{p} = c_n$.

[25] We are using the fact that each coloring in the sequence $c_p, c_{p+1}, c_{p+2}, \ldots$ extends the previous one, and so the union of this sequence is a function (coloring).

[26] Otherwise, we have that $A_k \subseteq [p]$, and we consider p instead of n.