NDMI011: Combinatorics and Graph Theory 1

Lecture #10

Sperner's theorem. Ramsey numbers

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1. Sperner's theorem;

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1. Sperner's theorem;
2. the Pigeonhole Principle;
3. Ramsey numbers.

Part I: Sperner's theorem

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Definition

For a set X,

- a *chain* in $(\mathscr{P}(X), \subseteq)$ is any set $\mathcal{C}$ of subsets of X such that for all $C_1, C_2 \in \mathcal{C}$, we have that either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.[a]
- a *maximal chain* in $(\mathscr{P}(X), \subseteq)$ is a chain in $(\mathscr{P}(X), \subseteq)$ such that there is no chain $\mathcal{C}'$ in $(\mathscr{P}(X), \subseteq)$ with the property that $\mathcal{C} \subsetneqq \mathcal{C}'$;
- an *antichain* in $(\mathscr{P}(X), \subseteq)$ is any set $\mathcal{A}$ of subsets of X such that for all distinct $A_1, A_2 \in \mathcal{A}$, we have that $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$.[b]

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[a] This definition works both for finite and for infinite X. Note also that $\emptyset$ is a chain in $(\mathscr{P}(X), \subseteq)$. However, if X is finite and $\mathcal{C}$ is a non-empty chain in $(\mathscr{P}(X), \subseteq)$, then $\mathcal{C}$ can be ordered as $\mathcal{C} = \{C_1, \dots, C_t\}$ so that $C_1 \subseteq \dots \subseteq C_t$.

[b] Equivalently: $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are both non-empty.

Example 2.1

Let $X = \{1, 2, 3, 4\}$. The following are chains in $(\mathscr{P}(X), \subseteq)$:[a]

- $\{\{2, 4\}, \{1, 2, 4\}\}$;[b]
- $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}$.[c]
- $\{\emptyset, \{4\}, \{2, 4\}, \{1, 2, 4\}, X\}$;[d]

Further, the following are all antichains in $(\mathscr{P}(X), \subseteq)$:[e]

- $\{\emptyset\}$;
- $\{X\}$;
- $\{\{1, 2\}, \{2, 3\}, \{1, 3, 4\}\}$;
- $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

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[a] There are many other chains in $(\mathscr{P}(X), \subseteq)$ as well.

[b] Note that this chain is **not** maximal, since we can add (for example) the set $\{2\}$ to it and obtain a larger chain.

[c] This chain is maximal.

[d] This chain is maximal.

[e] There are many other antichains in $(\mathscr{P}(X), \subseteq)$ as well.

Observation

Let X be any set. Then a chain and an antichain in $(\mathscr{P}(X), \subseteq)$ can have at most one element in common.

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Proof. Let $\mathcal{C}$ be a chain and $\mathcal{A}$ an antichain in $(\mathscr{P}(X), \subseteq)$, and suppose that $|\mathcal{C} \cap \mathcal{A}| \geq 2$. Fix distinct $X_1, X_2 \in \mathcal{C} \cap \mathcal{A}$. Since $X_1, X_2 \in \mathcal{C}$, we have that either $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$. But this is impossible, because X_1 and X_2 are distinct elements of the antichain $\mathcal{A}$.

Sperner's theorem

Let n be a non-negative integer, and let X be an n-element set. Then any antichain in $(\mathscr{P}(X), \subseteq)$ has at most $\binom{n}{\lfloor n/2 \rfloor}$ elements. Furthermore, this bound is tight, that is, there exists an antichain in $(\mathscr{P}(X), \subseteq)$ that has precisely $\binom{n}{\lfloor n/2 \rfloor}$ elements.

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Proof. First, we note that the set of all $\lfloor n/2 \rfloor$-element subsets of X is an antichain in $(\mathscr{P}(X), \subseteq)$, and this antichain has precisely $\binom{n}{\lfloor n/2 \rfloor}$ elements. It remains to show that any antichain in $(\mathscr{P}(X), \subseteq)$ has at most $\binom{n}{\lfloor n/2 \rfloor}$ elements.

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Proof (continued).

> ***Claim 1.*** *There are precisely $n!$ maximal chains in $(\mathscr{P}(X), \subseteq)$.*

Proof of Claim 1. Clearly, any maximal chain in $(\mathscr{P}(X), \subseteq)$ is of the form $\{\emptyset, \{x_1\}, \{x_1, x_2\}, \dots, \{x_1, x_2, \dots, x_n\}\}$, where $x_1, \dots, x_n$ is some ordering of the elements of X. There are precisely $n!$ such orderings, and so the number of maximal chains in $(\mathscr{P}(X), \subseteq)$ is $n!$. $\blacksquare$

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Proof (continued).

> ***Claim 2.*** *For every set $A \subseteq X$, the number of maximal chains of $(\mathscr{P}(X), \subseteq)$ containing A is precisely $|A|!(n-|A|)!$.*

Proof of Claim 2. Set $k = |A|$. Any chain in $(\mathscr{P}(X), \subseteq)$ is of the form $\{\emptyset, \{x_1\}, \{x_1, x_2\}, \dots, \{x_1, x_2, \dots, x_n\}\}$, where $x_1, \dots, x_n$ is some ordering of the elements of X; this chain contains A iff $A = \{x_1, \dots, x_k\}$ (and therefore, $X \setminus A = \{x_{k+1}, \dots, x_n\}$). The number of ways of ordering A is $k!$, and the number of ways of ordering $X \setminus A$ is $(n-k)!$. So, the total number of chains of $(\mathscr{P}(X), \subseteq)$ containing A is precisely $k!(n-k)!$. ■

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Proof (continued). Fix an antichain $\mathcal{A}$ in $(\mathscr{P}(X), \subseteq)$. We form the matrix M whose rows are indexed by the elements of $\mathcal{A}$, and whose columns are indexed by the maximal chains of $(\mathscr{P}(X), \subseteq)$, and in which the $(A, \mathcal{C})$-th entry is 1 if $A \in \mathcal{C}$ and is 0 otherwise.

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Our goal is to count the number of 1's in the matrix M in two ways.

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Proof (continued). First, by Claim 2, for any $A \in \mathcal{A}$, the number of maximal chains of $(\mathscr{P}(X), \subseteq)$ containing A is precisely $|A|!(n-|A|)!$; so, the number of 1's in the row of M indexed by A is precisely $|A|!(n-|A|)!$.

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$$\sum_{A \in \mathcal{A}} |A|!(n-|A|)!.$$

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On the other hand, by Claim 1, the number of columns of M is precisely $n!$. Furthermore, no chain of $(\mathscr{P}(X), \subseteq)$ contains more than one element of the antichain $\mathcal{A}$, and so no column of M contains more than one 1. So, the total number of 1's in the matrix M is at most $n!$.

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Proof (continued). We now have that $\sum\limits_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!$,

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Proof (continued). We now have that $\sum\limits_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!$, and consequently, $\sum\limits_{A \in \mathcal{A}} \frac{|A|!(n-|A|)!}{n!} \leq 1$.

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$$\frac{|A|!(n-|A|)!}{n!} = \frac{1}{\frac{n!}{|A|!(n-|A|)!}} = \frac{1}{\binom{n}{|A|}} \geq \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}.$$

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It follows that

$$1 \quad \geq \quad \sum_{A \in \mathcal{A}} \frac{|A|!(n-|A|)!}{n!} \quad \geq \quad \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \quad \geq \quad |\mathcal{A}| \frac{1}{\binom{n}{\lfloor n/2 \rfloor}},$$

and consequently, $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$. This completes the argument.

Part II: The Pigeonhole Principle

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The Pigeonhole Principle

Let $n_1, \dots, n_t$ ($t \geq 1$) be non-negative integers, and let X be a set of size at least $1 + n_1 + \dots + n_t$. If $(X_1, \dots, X_t)$ is any partition of X,[a] then there exists some $i \in \{1, \dots, t\}$ such that $|X_i| > n_i$.

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Proof. Suppose otherwise, and fix a partition $(X_1, \dots, X_t)$ such that $|X_i| \leq n_i$ for all $i \in \{1, \dots, t\}$. But then

$$\begin{aligned} 1 + n_1 + \cdots + n_t &\leq |X| \\ &= |X_1| + \cdots + |X_t| \\ &\leq n_1 + \cdots + n_t, \end{aligned}$$

a contradiction.

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Corollary 2.1

Let n and t be positive integers. Let X be an n-element set, and let $(X_1, \ldots, X_t)$ be any partition of X.[a] Then there exists some $i \in \{1, \ldots, t\}$ such that $|X_i| \geq \lceil \frac{n}{t} \rceil$.

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Proof. Lecture Notes.

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- Remark: Corollary 2.1 itself is sometimes referred to as the Pigeonhole Principle.

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A *stable set* (or *independent set*) in a graph G is any set of pairwise non-adjacent vertices of G. The *stability number* (or *independence number*) of G, denoted by $\alpha(G)$, is the maximum size of a stable set in G.

Proposition 3.1

Let G be a graph on at least six vertices. Then either $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.

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Proof. Let u be any vertex of G. Then $|V(G) \setminus \{u\}| \geq 5$, and so (by the Pigeonhole Principle) either u has at least three neighbors or it has at least three non-neighbors.

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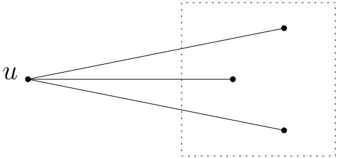
Suppose first that u has at least three neighbors.



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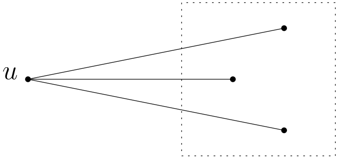

If at least two of those neighbors, say u_1 and u_2, are adjacent, then $\{u, u_1, u_2\}$ is a clique of G of size three, and we deduce that $\omega(G) \geq 3$.

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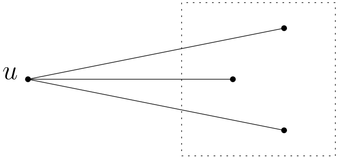

If at least two of those neighbors, say u_1 and u_2, are adjacent, then $\{u, u_1, u_2\}$ is a clique of G of size three, and we deduce that $\omega(G) \geq 3$. On the other hand, if no two neighbors of u are adjacent, then they together form a stable set of size at least three, and we deduce that $\alpha(G) \geq 3$.

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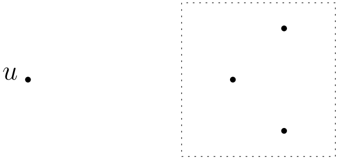
Proof (continued). Suppose now that u has at least three non-neighbors.



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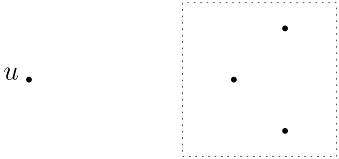

If at least two of those non-neighbors, say u_1 and u_2, are non-adjacent, then $\{u, u_1, u_2\}$ is a stable set of G of size three, and we deduce that $\alpha(G) \geq 3$.

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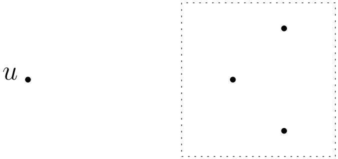

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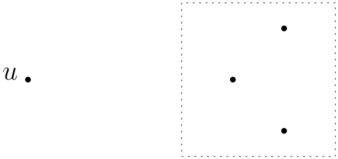

If at least two of those non-neighbors, say u_1 and u_2, are non-adjacent, then $\{u, u_1, u_2\}$ is a stable set of G of size three, and we deduce that $\alpha(G) \geq 3$. On the other hand, if the non-neighbors of u are pairwise adjacent, then they together form a clique of size at least three, and we deduce that $\omega(G) \geq 3$.

Theorem 3.2

Let k and ℓ be positive integers, and let G be a graph on at least $\binom{k+\ell-2}{k-1}$ vertices. Then either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

Proof.

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Let k and ℓ be positive integers, and let G be a graph on at least $\binom{k+\ell-2}{k-1}$ vertices. Then either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

Proof. We may assume inductively that for all positive integers k', ℓ' such that $k' + \ell' < k + \ell$, all graphs G' on at least $\binom{k'+\ell'-2}{k'-1}$ vertices satisfy either $\omega(G') \geq k'$ or $\alpha(G') \geq \ell'$.

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If $k = 1$ or $\ell = 1$, then the result is immediate. So, we may assume that $k, \ell \geq 2$.

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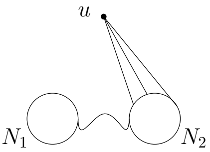
Let k and ℓ be positive integers, and let G be a graph on at least $\binom{k+\ell-2}{k-1}$ vertices. Then either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

Proof. We may assume inductively that for all positive integers k', ℓ' such that $k' + \ell' < k + \ell$, all graphs G' on at least $\binom{k'+\ell'-2}{k'-1}$ vertices satisfy either $\omega(G') \geq k'$ or $\alpha(G') \geq \ell'$.

If $k = 1$ or $\ell = 1$, then the result is immediate. So, we may assume that $k, \ell \geq 2$.

Now, set $n = \binom{k+\ell-2}{k-1}$, $n_1 = \binom{k+\ell-3}{k-1}$, and $n_2 = \binom{k+\ell-3}{k-2}$; then $n = n_1 + n_2$, and consequently, $n - 1 = 1 + (n_1 - 1) + (n_2 - 1)$.

Fix any vertex $u \in V(G)$, and set $N_1 = V(G) \setminus N_G[u]$ and $N_2 = N_G(u)$.



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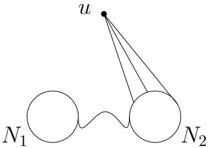

Since (N_1, N_2) is a partition of $V(G) \setminus \{u\}$, and since

$$|V(G) \setminus \{u\}| \quad \geq \quad n - 1 \quad = \quad 1 + (n_1 - 1) + (n_2 - 1),$$

the Pigeonhole Principle guarantees that either $|N_1| \geq n_1$ or $|N_2| \geq n_2$.

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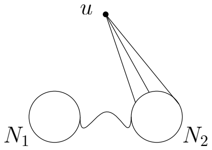

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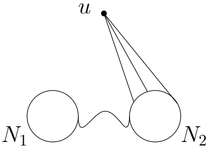

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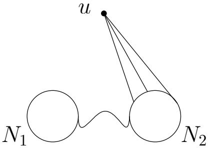

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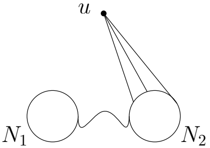

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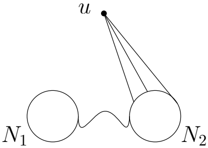

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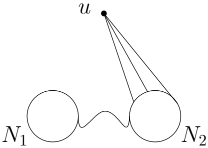

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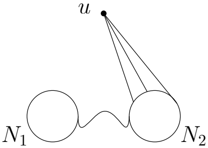

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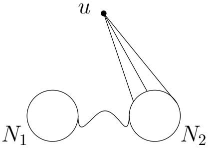

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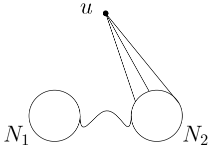

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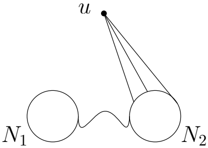

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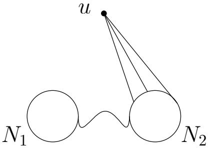

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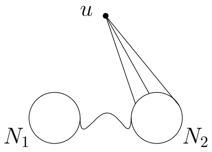

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- Numbers $R(k, \ell)$ (with $k, \ell \geq 1$) are called *Ramsey numbers*.

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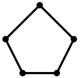

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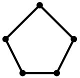

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- The exact values of a few other Ramsey numbers are known, but no general formula for $R(k, \ell)$ is known.
- Note however, that Theorem 3.2 gives an upper bound for Ramsey numbers, namely, $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$ for all $k, \ell \geq 1$.

Theorem 3.3

For all integers $k \geq 3$, we have that $R(k,k) > 2^{k/2}$.

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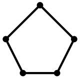
Proof. Since $\omega(C_5) = 2$ and $\alpha(C_5) = 2$, we see that $R(3,3) > 5 > 2^{3/2}$ and $R(4,4) > 5 > 2^{4/2}$. Thus, the claim holds for $k = 3$ and $k = 4$.



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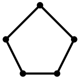

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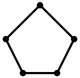

From now on, we assume that $k \geq 5$.

Let G be a graph on $n := \lfloor 2^{k/2} \rfloor$ vertices, with adjacency as follows: between any two distinct vertices, we (independently) put an edge with probability $\frac{1}{2}$ (and a non-edge with probability $\frac{1}{2}$).

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For all integers $k \geq 3$, we have that $R(k,k) > 2^{k/2}$.

Proof (continued). Thus, the probability that G satisfies at least one of $\omega(G) \geq k$ and $\alpha(G) \geq k$ is at most

$$\begin{array}{rcll} 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} & \leq & 2\left(\frac{en}{k}\right)^k\left(\frac{1}{2}\right)^{\binom{k}{2}} & \text{by Theorem 2.1} \\ & & & \text{from Lecture Notes 1} \\ & \leq & \dfrac{2\left(\frac{e2^{k/2}}{k}\right)^k}{2^{k(k-1)/2}} & \text{because } n = \lfloor 2^{k/2} \rfloor \\ & = & 2\left(\dfrac{e2^{k/2}}{k2^{(k-1)/2}}\right)^k & \\ & < & 2\left(\dfrac{e\sqrt{2}}{k}\right)^k & \\ & < & 1 & \text{because } k \geq 5 \end{array}$$

Theorem 3.3

For all integers $k \geq 3$, we have that $R(k, k) > 2^{k/2}$.

Proof (continued). Thus, the probability that G satisfies neither $\omega(G) \geq k$ nor $\alpha(G) \geq k$ is strictly positive.

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This proves that $R(k, k) > \lfloor 2^{k/2} \rfloor$; since $R(k, k)$ is an integer, we deduce that $R(k, k) > 2^{k/2}$.