NDMI011: Combinatorics and Graph Theory 1

Lecture #10
Sperner’s theorem. Ramsey numbers

Irena Penev

1 Sperner’s theorem

For a partially ordered set $(X, \leq)$,

- a chain in $(X, \leq)$ is any set $C \subseteq X$ such that for all $x_1, x_2 \in C$, we have that either $x_1 \leq x_2$ or $x_2 \leq x_1$.\(^1\)

- a maximal chain in $(X, \leq)$ is a chain $C$ in $(X, \leq)$ such that there is no chain $C'$ in $(X, \leq)$ with the property that $C \subsetneq C'$;

- an antichain in $(X, \leq)$ is any set $A \subseteq X$ such that for all distinct $x_1, x_2 \in A$, we have that $x_1 \not\leq x_2$ and $x_2 \not\leq x_1$.

Note that a chain and an antichain in $(X, \leq)$ can have at most one element in common.\(^2\)

Here, we are interested in a special case of the above. As usual, for a set $X$, we denote by $\mathcal{P}(X)$ the power set (i.e. the set of all subsets) of $X$. Clearly, for any set $X$, $\subseteq \mathcal{P}(X) := \{(A, B) \mid A, B \in \mathcal{P}(X), A \subseteq B\}$ is a partial order on $X$. To simplify notation, in what follows, we write $(\mathcal{P}(X), \subseteq)$ instead of $(\mathcal{P}(X), \subseteq \mathcal{P}(X))$. We apply the above definitions to $(\mathcal{P}(X), \subseteq)$, as follows.

For a set $X$,

\(^1\)This definition works both for finite and for infinite $X$. Note also that $\emptyset$ is a chain in $(X, \leq)$. However, if $X$ is finite and $C$ is a non-empty chain in $(X, \leq)$, then $C$ can be ordered as $C = \{x_1, \ldots, x_t\}$ so that $x_1 \leq \cdots \leq x_t$.

\(^2\)Indeed, if distinct elements $x_1, x_2$ belong to a chain of $(X, \leq)$, then $x_1 \leq x_2$ or $x_2 \leq x_1$. On the other hand, if they belong to an antichain of $(X, \leq)$, then $x_1 \not\leq x_2$ and $x_2 \not\leq x_1$. So, distinct elements $x_1$ and $x_2$ cannot simultaneously belong to a chain and an antichain of $(X, \leq)$.
• a chain in \((\mathcal{P}(X), \subseteq)\) is any set \(C\) of subsets of \(X\) such that for all \(C_1, C_2 \in C\), we have that either \(C_1 \subseteq C_2\) or \(C_2 \subseteq C_1\).\(^3\)

• a maximal chain in \((\mathcal{P}(X), \subseteq)\) is a chain in \((\mathcal{P}(X), \subseteq)\) such that there is no chain \(C'\) in \((\mathcal{P}(X), \subseteq)\) with the property that \(C \subseteq C'\);

• an antichain in \((\mathcal{P}(X), \subseteq)\) is any set \(A\) of subsets of \(X\) such that for all distinct \(A_1, A_2 \in A\), we have that \(A_1 \nsubseteq A_2\) and \(A_2 \nsubseteq A_1\).\(^4\)

As before, note that a chain and an antichain in \((\mathcal{P}(X), \subseteq)\) can have at most one element in common.

**Example 1.1.** Let \(X = \{1, 2, 3, 4\}\). The following are chains in \((\mathcal{P}(X), \subseteq)\).\(^5\)

- \(\{\{2, 4\}, \{1, 2, 4\}\}\).\(^6\)
- \(\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}\).\(^7\)
- \(\{\emptyset, \{4\}, \{2, 4\}, \{1, 2, 4\}, X\}\).\(^8\)

Further, the following are all antichains in \((\mathcal{P}(X), \subseteq)\).\(^9\)

- \(\{\emptyset\}\);
- \(\{X\}\);
- \(\{\{1, 2\}, \{2, 3\}, \{1, 3, 4\}\}\);
- \(\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}\).

**Sperner’s theorem.** Let \(n\) be a non-negative integer, and let \(X\) be an \(n\)-element set. Then any antichain in \((\mathcal{P}(X), \subseteq)\) has at most \(\binom{n}{\lfloor n/2 \rfloor}\) elements. Furthermore, this bound is tight, that is, there exists an antichain in \((\mathcal{P}(X), \subseteq)\) that has precisely \(\binom{n}{\lfloor n/2 \rfloor}\) elements.

**Proof.** First, we note that the set of all \([n/2]\)-element subsets of \(X\) is an antichain in \((\mathcal{P}(X), \subseteq)\), and this antichain has precisely \(\binom{n}{\lfloor n/2 \rfloor}\) elements. It remains to show that any antichain in \((\mathcal{P}(X), \subseteq)\) has at most \(\binom{n}{\lfloor n/2 \rfloor}\) elements.

\(^3\)This definition works both for finite and for infinite \(X\). Note also that \(\emptyset\) is a chain in \((\mathcal{P}(X), \subseteq)\). However, if \(X\) is finite and \(C\) is a non-empty chain in \((\mathcal{P}(X), \subseteq)\), then \(C\) can be ordered as \(C = \{C_1, \ldots, C_t\}\) so that \(C_1 \subseteq \cdots \subseteq C_t\).

\(^4\)Equivalently: \(A_1 \setminus A_2\) and \(A_2 \setminus A_1\) are both non-empty.

\(^5\)There are many other chains in \((\mathcal{P}(X), \subseteq)\) as well.

\(^6\)Note that this chain is not maximal, since we can add (for example) the set \(\{2\}\) to it and obtain a larger chain.

\(^7\)This chain is maximal.

\(^8\)This chain is maximal.

\(^9\)There are many other antichains in \((\mathcal{P}(X), \subseteq)\) as well.
Claim 1. There are precisely \( n! \) maximal chains in \( (\mathcal{P}(X), \subseteq) \).

*Proof of Claim 1.* Clearly, any maximal chain in \( (\mathcal{P}(X), \subseteq) \) is of the form \( \emptyset, \{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, x_2, \ldots, x_n\} \), where \( x_1, \ldots, x_n \) is some ordering of the elements of \( X \). There are precisely \( n! \) such orderings, and so the number of maximal chains in \( (\mathcal{P}(X), \subseteq) \) is \( n! \). ■

Claim 2. For every set \( A \subseteq X \), the number of maximal chains of \( (\mathcal{P}(X), \subseteq) \) containing \( A \) is precisely \( |A|!(n - |A|)! \).

*Proof of Claim 2.* Set \( k = |A| \). As in the proof of Claim 1, we have that any chain in \( (\mathcal{P}(X), \subseteq) \) is of the form \( \emptyset, \{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, x_2, \ldots, x_n\} \), where \( x_1, \ldots, x_n \) is some ordering of the elements of \( X \); this chain contains \( A \) if and only if \( A = \{x_1, \ldots, x_k\} \) (and therefore, \( X \setminus A = \{x_{k+1}, \ldots, x_n\} \)). The number of ways of ordering \( X \setminus A \) is \( k! \), and the number of ways of ordering \( X \setminus A \) is \( (n - k)! \). So, the total number of chains of \( (\mathcal{P}(X), \subseteq) \) containing \( A \) is precisely \( k!(n-k)! \). ■

Now, fix an antichain \( A \) in \( (\mathcal{P}(X), \subseteq) \). We form the matrix \( M \) whose rows are indexed by the elements of \( A \), and whose columns are indexed by the maximal chains of \( (\mathcal{P}(X), \subseteq) \), and in which the \((A,C)\)-th entry is 1 if \( A \in C \) and is 0 otherwise.\(^{10}\) Our goal is to count the number of 1’s in the matrix \( M \) in two ways.

First, by Claim 2, for any \( A \in \mathcal{A} \), the number of maximal chains of \( (\mathcal{P}(X), \subseteq) \) containing \( A \) is precisely \( |A|!(n - |A|)! \); so, the total number of 1’s in the matrix \( M \) is precisely

\[
\sum_{A \in \mathcal{A}} |A|!(n - |A|)!.
\]

On the other hand, by Claim 1, the number of columns of \( M \) is precisely \( n! \). Furthermore, no chain of \( (\mathcal{P}(X), \subseteq) \) contains more than one element of the antichain \( \mathcal{A} \), and so no column of \( M \) contains more than one 1. So, the total number of 1’s in the matrix \( M \) is at most \( n! \). We now have that

\[
\sum_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!,
\]

and consequently,

\[
\sum_{A \in \mathcal{A}} \frac{|A|!(n - |A|)!}{n!} \leq 1.
\]

On the other hand, for all \( A \subseteq X \) (and in particular, for all \( A \in \mathcal{A} \)), we have that

\[
\frac{|A|!(n - |A|)!}{n!} = \frac{1}{\binom{n}{|A|}} \cdot \binom{n}{\lfloor n/2 \rfloor} \geq \frac{1}{\binom{n}{\lfloor n/2 \rfloor}},
\]

where (*) follows from the fact that \( \binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor} \) for all \( k \in \{0, \ldots, n\} \).\(^{11}\)

\(^{10}\)Here, \( A \in \mathcal{A}, \mathcal{C} \) is a maximal chain in \( (\mathcal{P}(X), \subseteq) \), and the \((A,C)\)-th entry of \( M \) is the entry in the row indexed by \( A \) and column indexed by \( C \).

\(^{11}\)See subsection 2.2 of Lecture Notes 1.
We now have that

\[ 1 \geq \sum_{A \in \mathcal{A}} \frac{|A!(n-|A|)!}{n!} \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \geq |\mathcal{A}| \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}, \]

which yields \( |\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor} \). This completes the argument.

\[ \square \]

## 2 The Pigeonhole Principle

**The Pigeonhole Principle.** Let \( n_1, \ldots, n_t \ (t \geq 1) \) be non-negative integers, and let \( X \) be a set of size at least \( 1 + n_1 + \cdots + n_t \). If \( (X_1, \ldots, X_t) \) is any partition of \( X \), then there exists some \( i \in \{1, \ldots, t\} \) such that \( |X_i| > n_i \). \(^{12}\)

**Proof.** Suppose otherwise, and fix a partition \((X_1, \ldots, X_t)\) such that \( |X_i| \leq n_i \) for all \( i \in \{1, \ldots, t\} \). But then

\[ 1 + n_1 + \cdots + n_t \leq |X| = |X_1| + \cdots + |X_t| \leq n_1 + \cdots + n_t, \]

a contradiction. \( \square \)

As an immediate corollary, we obtain the following.

**Corollary 2.1.** Let \( n \) and \( t \) be positive integers. Let \( X \) be an \( n \)-element set, and let \( (X_1, \ldots, X_t) \) be any partition of \( X \). \(^{14}\) Then there exists some \( i \in \{1, \ldots, t\} \) such that \( |X_i| \geq \lceil \frac{n}{t} \rceil \).

**Proof.** By the Pigeonhole Principle, we need only show that \( n \geq 1 + t(\lceil \frac{n}{t} \rceil - 1) \).

If \( t \mid n, \) then \( \lceil \frac{n}{t} \rceil = \frac{n}{t} \), and we have that

\[ 1 + t(\lceil \frac{n}{t} \rceil - 1) \leq 1 + t(\frac{n}{t} - 1) = n - t + 1 \leq n, \]

which is what we needed. Suppose now that \( t \not\mid n \), so that \( \lceil \frac{n}{t} \rceil - 1 = \lfloor \frac{n}{t} \rfloor \). Then let \( m = \lfloor \frac{n}{t} \rfloor \) and \( \ell = n - mt \); since \( t \not\mid n \), we have that \( \ell \geq 1 \). But now

\[ 1 + t(\lfloor \frac{n}{t} \rfloor - 1) = 1 + t(\lfloor \frac{n}{t} \rfloor) = 1 + tm \leq \ell + tm \leq n, \]

and we are done. \( \square \)

We remark that Corollary 2.1 is also often referred to as the Pigeonhole Principle.

\(^{12}\)Here, we allow the sets \( X_1, \ldots, X_t \) to possibly be empty.

\(^{13}\)If one thinks of elements of \( X \) as “pigeons” and sets \( X_1, \ldots, X_t \) as “pigeonholes,” then the Pigeonhole Principle states that some pigeonhole \( X_i \) receives more than \( n_i \) pigeons.

\(^{14}\)Here, we allow the sets \( X_1, \ldots, X_t \) to possibly be empty.

\(^{15}\)“\( t \mid n \)” means that \( n \) is divisible by \( t \).
3 Ramsey numbers

A clique in a graph $G$ is any set of pairwise adjacent vertices of $G$. The clique number of $G$, denoted by $\omega(G)$, is the maximum size of a clique of $G$.

A stable set (or independent set) in a graph $G$ is any set of pairwise non-adjacent vertices of $G$. The stability number (or independence number) of $G$, denoted by $\alpha(G)$, is the maximum size of a stable set in $G$.

**Proposition 3.1.** Let $G$ be a graph on at least six vertices. Then either $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.

**Proof.** Let $u$ be any vertex of $G$. Then $|V(G) \setminus \{u\}| \geq 5$, and so (by the Pigeonhole Principle) either $u$ has at least three neighbors or it has at least three non-neighbors.

Suppose first that $u$ has at least three neighbors. If at least two of those neighbors, say $u_1$ and $u_2$, are adjacent, then $\{u, u_1, u_2\}$ is a clique of $G$ of size three, and we deduce that $\omega(G) \geq 3$. On the other hand, if no two neighbors of $u$ are adjacent, then they together form a stable set of size at least three, and we deduce that $\alpha(G) \geq 3$.

Suppose now that $u$ has at least three non-neighbors. If at least two of those non-neighbors, say $u_1$ and $u_2$, are non-adjacent, then $\{u, u_1, u_2\}$ is a stable set of $G$ of size three, and we deduce that $\alpha(G) \geq 3$. On the other hand, if the non-neighbors of $u$ are pairwise adjacent, then they together form a clique of size at least three, and we deduce that $\omega(G) \geq 3$. \hfill \square

For a graph $G$ and a vertex $u$, $N_G(u)$ is the set of all neighbors of $u$ in $G$, and $N_G[u] = \{u\} \cup N_G(u)$.

**Theorem 3.2.** Let $k$ and $\ell$ be positive integers, and let $G$ be a graph on at least $\binom{k+\ell-2}{k-1}$ vertices.\(^{16}\) Then either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

**Proof.** We may assume inductively that for all positive integers $k', \ell'$ such that $k' + \ell' < k + \ell$, all graphs $G'$ on at least $\binom{k'+\ell'-2}{k'-1}$ vertices satisfy either $\omega(G') \geq k'$ or $\alpha(G') \geq \ell'$.

If $k = 1$ or $\ell = 1$, then the result is immediate.\(^{17}\) So, we may assume that $k, \ell \geq 2$. Now, set $n = \binom{k+\ell-2}{k-1}$, $n_1 = \binom{k+\ell-3}{k-1}$, and $n_2 = \binom{k+\ell-3}{k-2}$; then $n = n_1 + n_2$, and consequently, $n - 1 = 1 + (n_1 - 1) + (n_2 - 1)$. Fix any vertex $u \in V(G)$, and set $N_1 = V(G) \setminus N_G[u]$ and $N_2 = N_G(u)$.

\(^{16}\)Note that $\binom{k+\ell-2}{k-1} = \binom{\ell-2}{\ell-1}$.

\(^{17}\)Indeed, it is clear that $\omega(G) \geq 1$ and $\alpha(G) \geq 1$. So, if $k = 1$, then $\omega(G) \geq k$; and if $\ell = 1$, then $\alpha(G) \geq \ell$. 

5
Since \((N_1, N_2)\) is a partition of \(V(G) \setminus \{u\}\), and since \(|V(G) \setminus \{u\}| \geq n - 1 = 1 + (n_1 - 1) + (n_2 - 1)\), the Pigeonhole Principle guarantees that either \(|N_1| \geq n_1\) or \(|N_2| \geq n_2\).

Suppose first that \(|N_1| \geq n_1\), i.e. \(|N_1| \geq \left(\frac{k+\ell-1-2}{k-1}\right)\). Then by the induction hypothesis, either \(\omega(G[N_1]) \geq k\) or \(\alpha(G[N_1]) \geq \ell - 1\). In the former case, we have that \(\omega(G) \geq \omega(G[N_1]) \geq k\), and we are done. So suppose that \(\alpha(G[N_1]) \geq \ell - 1\). Then let \(S\) be a stable set of \(G[N_1]\) of size \(\ell - 1\). Then \(\{u\} \cup S\) is a stable set of size \(\ell\) in \(G\), we deduce that \(\alpha(G) \geq \ell\), and again we are done.

Suppose now that \(|N_2| \geq n_2\), i.e. \(|N_2| \geq \left(\frac{(k-1)+\ell-2}{k-1}\right)\). Then by the induction hypothesis, either \(\omega(G[N_2]) \geq k - 1\) or \(\alpha(G[N_2]) \geq \ell\). In the latter case, we have that \(\alpha(G) \geq \alpha(G[N_2]) \geq \ell\), and we are done. So suppose that \(\omega(G[N_2]) \geq k - 1\). Then let \(C\) be a clique of \(G[N_2]\) of size \(k - 1\). But then \(\{u\} \cup C\) is a clique of size \(k\) in \(G\), we deduce that \(\omega(G) \geq k\), and again we are done. \(\square\)

For positive integers \(k\) and \(\ell\), we denote by \(R(k, \ell)\) the smallest number \(n\) such that every graph \(G\) on at least \(n\) vertices satisfies either \(\omega(G) \geq k\) or \(\alpha(G) \geq \ell\). The existence of \(R(k, \ell)\) follows immediately from Theorem 3.2. Numbers \(R(k, \ell)\) (with \(k, \ell \geq 1\)) are called Ramsey numbers.

It is easy to see that for all \(k, \ell \geq 1\), we have that\(^{18}\)

\[
R(1, \ell) = 1 \quad R(k, 1) = 1
\]

\[
R(2, \ell) = \ell \quad R(k, 2) = k
\]

Furthermore, we have \(R(3, 3) = 6\). Indeed, by Proposition 3.1, \(R(3, 3) \leq 6\). On the other hand, \(\omega(C_5) = 2\) and \(\alpha(C_5) = 2\), and so \(R(3, 3) > 5\). Thus, \(R(3, 3) = 6\). The exact values of a few other Ramsey numbers are known,\(^{19}\) but no general formula for \(R(k, \ell)\) is known. Note however, that Theorem 3.2 gives an upper bound for Ramsey numbers, namely, \(R(k, \ell) \leq \left(\frac{k+\ell-2}{k-1}\right)\) for all \(k, \ell \geq 1\).

We complete this section by giving a lower bound for the Ramsey number \(R(k, k)\).

**Theorem 3.3.** For all integers \(k \geq 3\), we have that \(R(k, k) > 2^{k/2}\).

**Proof.** Since \(\omega(C_5) = 2\) and \(\alpha(C_5) = 2\), we see that \(R(3, 3) > 5 > 2^{3/2}\) and \(R(4, 4) > 5 > 2^{4/2}\). Thus, the claim holds for \(k = 3\) and \(k = 4\). From now on, we assume that \(k \geq 5\).

Let \(G\) be a graph on \(n := \left\lfloor 2^{k/2} \right\rfloor\) vertices, with adjacency as follows: between any two distinct vertices, we (independently) put an edge with probability \(\frac{1}{2}\) (and a non-edge with probability \(\frac{1}{2}\)).

\(^{18}\)Check this!

\(^{19}\)For example, it is known that \(R(4, 4) = 18\). On the other hand, the exact value of \(R(5, 5)\) is still unknown.
For any set of $k$ vertices of $G$, the probability that this set is a clique is \( \binom{k}{2} \); there are \( \binom{n}{k} \) subsets of $V(G)$ of size $k$, and the probability that at least one of them is a clique is at most \( \binom{n}{k} \binom{k}{2} \). So, the probability that \( \omega(G) \geq k \) is at most \( \binom{n}{k} \binom{k}{2} \). Similarly, the probability that \( \alpha(G) \geq k \) is at most \( \binom{n}{k} \binom{k}{2} \). Thus, the probability that $G$ satisfies at least one of \( \omega(G) \geq k \) and \( \alpha(G) \geq k \) is at most

\[
2 \binom{n}{k} \binom{k}{2} \leq 2 \left( \frac{n}{k} \right)^k \binom{k}{2} \leq \frac{2^{e^{2k/2}}}{2^{k(k-1)/2}} \leq 2 \left( \frac{e^{2k/2}}{k^{2(k-1)/2}} \right)^k \leq 2 \left( \frac{\sqrt{2}}{k} \right)^k < 1
\]

by Theorem 2.1 from Lecture Notes 1, because $k \geq 5$.

Thus, the probability that $G$ satisfies neither $\omega(G) \geq k$ nor $\alpha(G) \geq k$ is strictly positive. So, there must be at least one graph on $n = \lfloor 2^{k/2} \rfloor$ vertices whose clique number and stability number are both strictly less than $k$. This proves that $R(k, k) > \lfloor 2^{k/2} \rfloor$; since $R(k, k)$ is an integer, we deduce that $R(k, k) > 2^{k/2}$.