NDMI011: Combinatorics and Graph Theory 1

Lecture #10
Sperner's theorem. Ramsey numbers

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1 Sperner's theorem

For a partially ordered set $(X, \leq)$,

- a *chain* in $(X, \leq)$ is any set $\mathcal{C} \subseteq X$ such that for all $x_1, x_2 \in \mathcal{C}$, we have that either $x_1 \leq x_2$ or $x_2 \leq x_1$.[1]
- a *maximal chain* in $(X, \leq)$ is a chain $\mathcal{C}$ in $(X, \leq)$ such that there is no chain $\mathcal{C}'$ in $(X, \leq)$ with the property that $\mathcal{C} \subsetneqq \mathcal{C}'$;
- an *antichain* in $(X, \leq)$ is any set $\mathcal{A} \subseteq X$ such that for all distinct $x_1, x_2 \in \mathcal{A}$, we have that $x_1 \not\leq x_2$ and $x_2 \not\leq x_1$.

Note that a chain and an antichain in $(X, \leq)$ can have at most one element in common.[2]

Here, we are interested in a special case of the above. As usual, for a set X, we denote by $\mathscr{P}(X)$ the power set (i.e. the set of all subsets) of X. Clearly, for any set X, $\subseteq_{\mathscr{P}(X)} := \{(A, B) \mid A, B \in \mathscr{P}(X), A \subseteq B\}$ is a partial order on X. To simplify notation, in what follows, we write $(\mathscr{P}(X), \subseteq)$ instead of $(\mathscr{P}(X), \subseteq_{\mathscr{P}(X)})$. We apply the above definitions to $(\mathscr{P}(X), \subseteq)$, as follows.

For a set X,

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[1] This definition works both for finite and for infinite X. Note also that $\emptyset$ is a chain in $(X, \leq)$. However, if X is finite and $\mathcal{C}$ is a non-empty chain in $(X, \leq)$, then $\mathcal{C}$ can be ordered as $\mathcal{C} = \{x_1, \dots, x_t\}$ so that $x_1 \leq \dots \leq x_t$.

[2] Indeed, if distinct elements x_1, x_2 belong to a chain of $(X, \leq)$, then $x_1 \leq x_2$ or $x_2 \leq x_1$. On the other hand, if they belong to an antichain of $(X, \leq)$, then $x_1 \not\leq x_2$ and $x_2 \not\leq x_1$. So, distinct elements x_1 and x_2 cannot simultaneously belong to a chain and an antichain of $(X, \leq)$.

- a *chain* in $(\mathscr{P}(X), \subseteq)$ is any set $\mathcal{C}$ of subsets of X such that for all $C_1, C_2 \in \mathcal{C}$, we have that either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.[3]
- a *maximal chain* in $(\mathscr{P}(X), \subseteq)$ is a chain in $(\mathscr{P}(X), \subseteq)$ such that there is no chain $\mathcal{C}'$ in $(\mathscr{P}(X), \subseteq)$ with the property that $\mathcal{C} \subsetneqq \mathcal{C}'$;
- an *antichain* in $(\mathscr{P}(X), \subseteq)$ is any set $\mathcal{A}$ of subsets of X such that for all distinct $A_1, A_2 \in \mathcal{A}$, we have that $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$.[4]

As before, note that a chain and an antichain in $(\mathscr{P}(X), \subseteq)$ can have at most one element in common.

Example 1.1. *Let* $X = \{1, 2, 3, 4\}$*. The following are chains in* $(\mathscr{P}(X), \subseteq)$*:*[5]

- $\{\{2, 4\}, \{1, 2, 4\}\}$;[6]
- $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}$.[7]
- $\{\emptyset, \{4\}, \{2, 4\}, \{1, 2, 4\}, X\}$;[8]

Further, the following are all antichains in $(\mathscr{P}(X), \subseteq)$*:*[9]

- $\{\emptyset\}$;
- $\{X\}$;
- $\{\{1, 2\}, \{2, 3\}, \{1, 3, 4\}\}$;
- $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

Sperner's theorem. *Let n be a non-negative integer, and let X be an n-element set. Then any antichain in $(\mathscr{P}(X), \subseteq)$ has at most $\binom{n}{\lfloor n/2 \rfloor}$ elements. Furthermore, this bound is tight, that is, there exists an antichain in $(\mathscr{P}(X), \subseteq)$ that has precisely $\binom{n}{\lfloor n/2 \rfloor}$ elements.*

Proof. First, we note that the set of all $\lfloor n/2 \rfloor$-element subsets of X is an antichain in $(\mathscr{P}(X), \subseteq)$, and this antichain has precisely $\binom{n}{\lfloor n/2 \rfloor}$ elements. It remains to show that any antichain in $(\mathscr{P}(X), \subseteq)$ has at most $\binom{n}{\lfloor n/2 \rfloor}$ elements.

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[3] This definition works both for finite and for infinite X. Note also that $\emptyset$ is a chain in $(\mathscr{P}(X), \subseteq)$. However, if X is finite and $\mathcal{C}$ is a non-empty chain in $(\mathscr{P}(X), \subseteq)$, then $\mathcal{C}$ can be ordered as $\mathcal{C} = \{C_1, \dots, C_t\}$ so that $C_1 \subseteq \dots \subseteq C_t$.

[4] Equivalently: $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are both non-empty.

[5] There are many other chains in $(\mathscr{P}(X), \subseteq)$ as well.

[6] Note that this chain is **not** maximal, since we can add (for example) the set $\{2\}$ to it and obtain a larger chain.

[7] This chain is maximal.

[8] This chain is maximal.

[9] There are many other antichains in $(\mathscr{P}(X), \subseteq)$ as well.

Claim 1. There are precisely $n!$ maximal chains in $(\mathscr{P}(X), \subseteq)$.

Proof of Claim 1. Clearly, any maximal chain in $(\mathscr{P}(X), \subseteq)$ is of the form $\{\emptyset, \{x_1\}, \{x_1, x_2\}, \dots, \{x_1, x_2, \dots, x_n\}\}$, where $x_1, \dots, x_n$ is some ordering of the elements of X. There are precisely $n!$ such orderings, and so the number of maximal chains in $(\mathscr{P}(X), \subseteq)$ is $n!$. ■

Claim 2. For every set $A \subseteq X$, the number of maximal chains of $(\mathscr{P}(X), \subseteq)$ containing A is precisely $|A|!(n - |A|)!$.

Proof of Claim 2. Set $k = |A|$. As in the proof of Claim 1, we have that any chain in $(\mathscr{P}(X), \subseteq)$ is of the form $\{\emptyset, \{x_1\}, \{x_1, x_2\}, \dots, \{x_1, x_2, \dots, x_n\}\}$, where $x_1, \dots, x_n$ is some ordering of the elements of X; this chain contains A if and only if $A = \{x_1, \dots, x_k\}$ (and therefore, $X \setminus A = \{x_{k+1}, \dots, x_n\}$). The number of ways of ordering A is $k!$, and the number of ways of ordering $X \setminus A$ is $(n-k)!$. So, the total number of chains of $(\mathscr{P}(X), \subseteq)$ containing A is precisely $k!(n-k)!$. ■

Now, fix an antichain $\mathcal{A}$ in $(\mathscr{P}(X), \subseteq)$. We form the matrix M whose rows are indexed by the elements of $\mathcal{A}$, and whose columns are indexed by the maximal chains of $(\mathscr{P}(X), \subseteq)$, and in which the $(A, \mathcal{C})$-th entry is 1 if $A \in \mathcal{C}$ and is 0 otherwise.[10] Our goal is to count the number of 1's in the matrix M in two ways.

First, by Claim 2, for any $A \in \mathcal{A}$, the number of maximal chains of $(\mathscr{P}(X), \subseteq)$ containing A is precisely $|A|!(n - |A|)!$; so, the number of 1's in the row of M indexed by A is precisely $|A|!(n - |A|)!$. Thus, the number of 1's in the matrix M is precisely

$$\sum_{A \in \mathcal{A}} |A|!(n - |A|)!.$$

On the other hand, by Claim 1, the number of columns of M is precisely $n!$. Furthermore, no chain of $(\mathscr{P}(X), \subseteq)$ contains more than one element of the antichain $\mathcal{A}$, and so no column of M contains more than one 1. So, the total number of 1's in the matrix M is at most $n!$. We now have that

$$\sum_{A \in \mathcal{A}} |A|!(n - |A|)! \quad \leq \quad n!,$$

and consequently,

$$\sum_{A \in \mathcal{A}} \frac{|A|!(n-|A|)!}{n!} \quad \leq \quad 1.$$

On the other hand, for all $A \subseteq X$ (and in particular, for all $A \in \mathcal{A}$), we have that

$$\frac{|A|!(n-|A|)!}{n!} \quad = \quad \frac{1}{\frac{n!}{|A|!(n-|A|)!}} \quad = \quad \frac{1}{\binom{n}{|A|}} \quad \overset{(*)}{\geq} \quad \frac{1}{\binom{n}{\lfloor n/2 \rfloor}},$$

where (*) follows from the fact that $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$ for all $k \in \{0, \dots, n\}$.[11]

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[10] Here, $A \in \mathcal{A}$, $\mathcal{C}$ is a maximal chain in $(\mathscr{P}(X), \subseteq)$, and the $(A, \mathcal{C})$-th entry of M is the entry in the row indexed by A and column indexed by $\mathcal{C}$.

[11] See subsection 2.2 of Lecture Notes 1.

We now have that

$$1 \quad \geq \quad \sum_{A \in \mathcal{A}} \frac{|A|!(n-|A|)!}{n!} \quad \geq \quad \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \quad \geq \quad |\mathcal{A}| \frac{1}{\binom{n}{\lfloor n/2 \rfloor}},$$

which yields $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$. This completes the argument. □

2 The Pigeonhole principle

The Pigeonhole Principle. *Let $n_1, \dots, n_t$ ($t \geq 1$) be non-negative integers, and let X be a set of size at least $1 + n_1 + \dots + n_t$. If $(X_1, \dots, X_t)$ is any partition of X,[12] then there exists some $i \in \{1, \dots, t\}$ such that $|X_i| > n_i$.[13]*

Proof. Suppose otherwise, and fix a partition $(X_1, \dots, X_t)$ such that $|X_i| \leq n_i$ for all $i \in \{1, \dots, t\}$. But then

$$1 + n_1 + \dots + n_t \quad \leq \quad |X| \quad = \quad |X_1| + \dots + |X_t| \quad \leq \quad n_1 + \dots + n_t,$$

a contradiction. □

As an immediate corollary, we obtain the following.

Corollary 2.1. *Let n and t be positive integers. Let X be an n-element set, and let $(X_1, \dots, X_t)$ be any partition of X.[14] Then there exists some $i \in \{1, \dots, t\}$ such that $|X_i| \geq \lceil \frac{n}{t} \rceil$.*

Proof. By the Pigeonhole Principle, we need only show that $n \geq 1 + t(\lceil \frac{n}{t} \rceil - 1)$. If $t \mid n$,[15] then $\lceil \frac{n}{t} \rceil = \frac{n}{t}$, and we have that

$$1 + t(\lceil \tfrac{n}{t} \rceil - 1) \quad \leq \quad 1 + t(\tfrac{n}{t} - 1) \quad = \quad n - t + 1 \quad \leq \quad n,$$

which is what we needed. Suppose now that $t \not| n$, so that $\lceil \frac{n}{t} \rceil - 1 = \lfloor \frac{n}{t} \lfloor$. Then let $m = \lfloor \frac{n}{t} \rfloor$ and $\ell = n - mt$; since $t \not| n$, we have that $\ell \geq 1$. But now

$$1 + t(\lceil \tfrac{n}{t} \rceil - 1) \quad = \quad 1 + t(\lfloor \tfrac{n}{t} \rfloor) \quad = \quad 1 + tm \quad \leq \quad \ell + tm \quad \leq \quad n,$$

and we are done. □

We remark that Corollary 2.1 is also often referred to as the Pigeonhole Principle.

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[12] Here, we allow the sets $X_1, \dots, X_t$ to possibly be empty.

[13] If one thinks of elements of X as "pigeons" and sets $X_1, \dots, X_t$ as "pigeonholes," then the Pigeonhole Principle states that some pigeonhole X_i receives more than n_i pigeons.

[14] Here, we allow the sets $X_1, \dots, X_t$ to possibly be empty.

[15] "$t \mid n$" means that n is divisible by t.

3 Ramsey numbers

A *clique* in a graph G is any set of pairwise adjacent vertices of G. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique of G.

A *stable set* (or *independent set*) in a graph G is any set of pairwise non-adjacent vertices of G. The *stability number* (or *independence number*) of G, denoted by $\alpha(G)$, is the maximum size of a stable set in G.

Proposition 3.1. *Let G be a graph on at least six vertices. Then either $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.*

Proof. Let u be any vertex of G. Then $|V(G) \setminus \{u\}| \geq 5$, and so (by the Pigeonhole Principle) either u has at least three neighbors or it has at least three non-neighbors.

Suppose first that u has at least three neighbors. If at least two of those neighbors, say u_1 and u_2, are adjacent, then $\{u, u_1, u_2\}$ is a clique of G of size three, and we deduce that $\omega(G) \geq 3$. On the other hand, if no two neighbors of u are adjacent, then they together form a stable set of size at least three, and we deduce that $\alpha(G) \geq 3$.

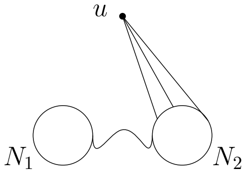
Suppose now that u has at least three non-neighbors. If at least two of those non-neighbors, say u_1 and u_2, are non-adjacent, then $\{u, u_1, u_2\}$ is a stable set of G of size three, and we deduce that $\alpha(G) \geq 3$. On the other hand, if the non-neighbors of u are pairwise adjacent, then they together form a clique of size at least three, and we deduce that $\omega(G) \geq 3$. $\square$

For a graph G and a vertex u, $N_G(u)$ is the set of all neighbors of u in G, and $N_G[u] = \{u\} \cup N_G(u)$.

Theorem 3.2. *Let k and ℓ be positive integers, and let G be a graph on at least $\binom{k+\ell-2}{k-1}$ vertices.*[16] *Then either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.*

Proof. We may assume inductively that for all positive integers k', ℓ' such that $k' + \ell' < k + \ell$, all graphs G' on at least $\binom{k'+\ell'-2}{k'-1}$ vertices satisfy either $\omega(G') \geq k'$ or $\alpha(G') \geq \ell'$.

If $k = 1$ or $\ell = 1$, then the result is immediate.[17] So, we may assume that $k, \ell \geq 2$. Now, set $n = \binom{k+\ell-2}{k-1}$, $n_1 = \binom{k+\ell-3}{k-1}$, and $n_2 = \binom{k+\ell-3}{k-2}$; then $n = n_1 + n_2$, and consequently, $n - 1 = 1 + (n_1 - 1) + (n_2 - 1)$. Fix any vertex $u \in V(G)$, and set $N_1 = V(G) \setminus N_G[u]$ and $N_2 = N_G(u)$.



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[16] Note that $\binom{k+\ell-2}{k-1} = \binom{k+\ell-2}{\ell-1}$.

[17] Indeed, it is clear that $\omega(G) \geq 1$ and $\alpha(G) \geq 1$. So, if $k = 1$, then $\omega(G) \geq k$; and if $\ell = 1$, then $\alpha(G) \geq \ell$.

Since (N_1, N_2) is a partition of $V(G) \setminus \{u\}$, and since $|V(G) \setminus \{u\}| \geq n-1 = 1 + (n_1 - 1) + (n_2 - 1)$, the Pigeonhole Principle guarantees that either $|N_1| \geq n_1$ or $|N_2| \geq n_2$.

Suppose first that $|N_1| \geq n_1$, i.e. $|N_1| \geq \binom{k+(\ell-1)-2}{k-1}$. Then by the induction hypothesis, either $\omega(G[N_1]) \geq k$ or $\alpha(G[N_1]) \geq \ell - 1$. In the former case, we have that $\omega(G) \geq \omega(G[N_1]) \geq k$, and we are done. So suppose that $\alpha(G[N_1]) \geq \ell - 1$. Then let S be a stable set of $G[N_1]$ of size $\ell - 1$. Then $\{u\} \cup S$ is a stable set of size ℓ in G, we deduce that $\alpha(G) \geq \ell$, and again we are done.

Suppose now that $|N_2| \geq n_2$, i.e. $|N_2| \geq \binom{(k-1)+\ell-2}{k-2}$. Then by the induction hypothesis, either $\omega(G[N_2]) \geq k-1$ or $\alpha(G[N_2]) \geq \ell$. In the latter case, we have that $\alpha(G) \geq \alpha(G[N_2]) \geq \ell$, and we are done. So suppose that $\omega(G[N_2]) \geq k-1$. Then let C be a clique of $G[N_2]$ of size $k-1$. But then $\{u\} \cup C$ is a clique of size k in G, we deduce that $\omega(G) \geq k$, and again we are done. □

For positive integers k and ℓ, we denote by $R(k, \ell)$ the smallest number n such that every graph G on at least n vertices satisfies either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$. The existence of $R(k, \ell)$ follows immediately from Theorem 3.2. Numbers $R(k, \ell)$ (with $k, \ell \geq 1$) are called *Ramsey numbers.*

It is easy to see that for all $k, \ell \geq 1$, we have that[18]

$$R(1,\ell) = 1 \qquad R(k,1) = 1$$

$$R(2,\ell) = \ell \qquad R(k,2) = k$$

Furthermore, we have $R(3,3) = 6$. Indeed, by Proposition 3.1, $R(3,3) \leq 6$. On the other hand, $\omega(C_5) = 2$ and $\alpha(C_5) = 2$, and so $R(3,3) > 5$. Thus, $R(3,3) = 6$. The exact values of a few other Ramsey numbers are known,[19] but no general formula for $R(k, \ell)$ is known. Note however, that Theorem 3.2 gives an upper bound for Ramsey numbers, namely, $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$ for all $k, \ell \geq 1$.

We complete this section by giving a lower bound for the Ramsey number $R(k, k)$.

Theorem 3.3. *For all integers $k \geq 3$, we have that $R(k, k) > 2^{k/2}$.*

Proof. Since $\omega(C_5) = 2$ and $\alpha(C_5) = 2$, we see that $R(3,3) > 5 > 2^{3/2}$ and $R(4,4) > 5 > 2^{4/2}$. Thus, the claim holds for $k = 3$ and $k = 4$. From now on, we assume that $k \geq 5$.

Let G be a graph on $n := \lfloor 2^{k/2} \rfloor$ vertices, with adjacency as follows: between any two distinct vertices, we (independently) put an edge with probability $\frac{1}{2}$ (and a non-edge with probability $\frac{1}{2}$).

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[18] Check this!

[19] For example, it is known that $R(4,4) = 18$. On the other hand, the exact value of $R(5,5)$ is still unknown.

For any set of k vertices of G, the probability that this set is a clique is $(\frac{1}{2})^{\binom{k}{2}}$; there are $\binom{n}{k}$ subsets of $V(G)$ of size k, and the probability that at least one of them is a clique is at most $\binom{n}{k}(\frac{1}{2})^{\binom{k}{2}}$. So, the probability that $\omega(G) \geq k$ is at most $\binom{n}{k}(\frac{1}{2})^{\binom{k}{2}}$. Similarly, the probability that $\alpha(G) \geq k$ is at most $\binom{n}{k}(\frac{1}{2})^{\binom{k}{2}}$. Thus, the probability that G satisfies at least one of $\omega(G) \geq k$ and $\alpha(G) \geq k$ is at most

$$\begin{aligned}
2\binom{n}{k}(\tfrac{1}{2})^{\binom{k}{2}} &\leq 2(\tfrac{en}{k})^k(\tfrac{1}{2})^{\binom{k}{2}} && \text{by Theorem 2.1} \\
& && \text{from Lecture Notes 1} \\
&\leq \frac{2(\frac{e2^{k/2}}{k})^k}{2^{k(k-1)/2}} && \text{because } n = \lfloor 2^{k/2} \rfloor \\
&= 2(\frac{e2^{k/2}}{k2^{(k-1)/2}})^k \\
&< 2(\frac{e\sqrt{2}}{k})^k \\
&< 1 && \text{because } k \geq 5
\end{aligned}$$

Thus, the probability that G satisfies neither $\omega(G) \geq k$ nor $\alpha(G) \geq k$ is strictly positive. So, there must be at least one graph on $n = \lfloor 2^{k/2} \rfloor$ vertices whose clique number and stability number are both strictly less than k. This proves that $R(k,k) > \lfloor 2^{k/2} \rfloor$; since $R(k,k)$ is an integer, we deduce that $R(k,k) > 2^{k/2}$. □