NDMI011: Combinatorics and Graph Theory 1

Lecture #9

2-connected graphs and the Ear lemma. Cayley's formula

Irena Penev

November 30, 2020

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1. the structure of 2-connected graphs (and the Ear lemma);

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2. the number of spanning trees of K_n (Cayley's formula).

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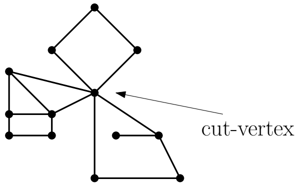

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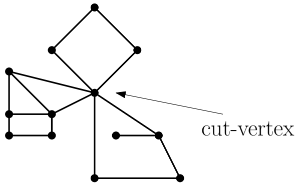

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For a non-negative integer k, a graph G is *k-connected* if $|V(G)| \geq k+1$ and for all $S \subseteq V(G)$ such that $|S| \leq k-1$, we have that $G \setminus S$ is connected.

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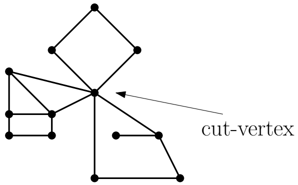

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- So, a graph is 2-connected if it has at least three vertices, is connected, and has no cut-vertices.

The global version of Menger's theorem

Let G be a graph on at least two vertices, and let $k, \ell \geq 0$ be integers.

(a) G is k-connected if and only if for all distinct $s, t \in V(G)$, there are k pairwise internally disjoint s-t paths in G.

(b) G is ℓ-edge-connected if and only if for all distinct $s, t \in E(G)$, there are ℓ pairwise edge-disjoint s-t paths in G.

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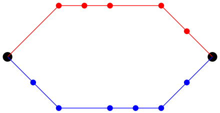
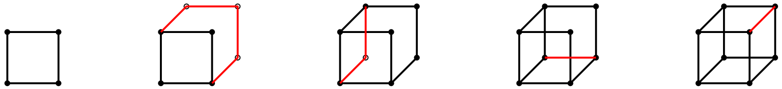
Proof. By Menger's theorem (global version), a graph on at least two vertices is 2-connected if and only if for any pair of distinct vertices, there are two internally disjoint paths between them. But obviously, two distinct vertices lie on a common cycle if and only if there are two internally-disjoint paths between them. The result now follows.



Definition

A *path addition* (sometimes called *ear addition*) to a graph H is the addition to H of a path between two distinct vertices of H in such a way that no internal vertex and no edge of the path belongs to H.



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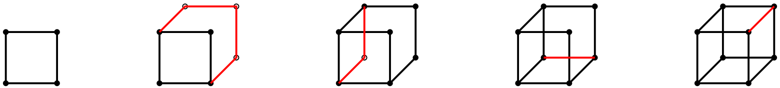

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Proof of the "$\Longleftarrow$" part. Clearly, cycles are 2-connected (indeed, every cycle has at least three vertices, is connected, and has no cut-vertices). Further, if a graph G can be obtained from a 2-connected graph H by adding an ear, then G has at least three vertices (because H does), and it is easy to see that G is connected and has no cut-vertices; so, G is 2-connected.

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Proof of the "$\Longrightarrow$" part (outline). Fix a 2-connected graph G. By Lemma 1.1, G contains a cycle. Now, let H be a maximal subgraph of G that either is a cycle or can be obtained from a cycle by repeated ear addition. We must show that $H = G$.

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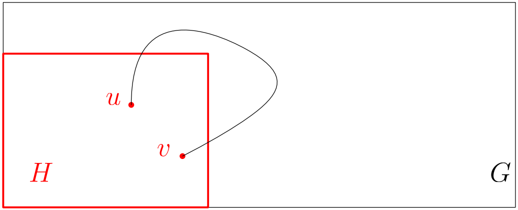
H is an induced subgraph of G, because otherwise, we can add another ear to H.



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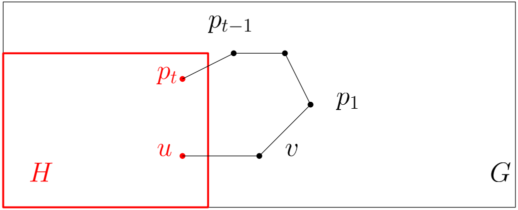

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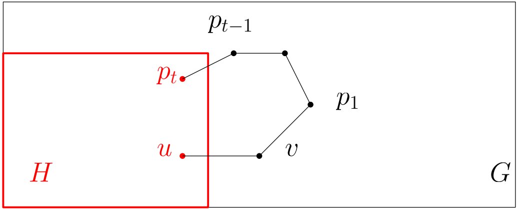


We now have that $V(H) = V(G)$, and that H is an induced subgraph of G. So, $H = G$.

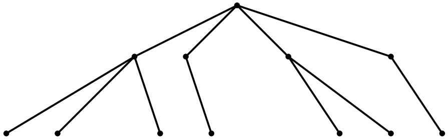
Part II: Cayley's formula

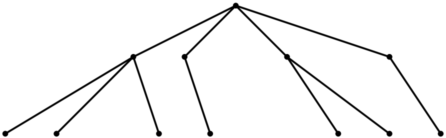
Definition

A *forest* is an acyclic graph (i.e. a graph that has no cycles), and a *tree* is a connected forest.

Definition

A *leaf* in a graph G is a vertex of degree one, i.e. a vertex that has exactly one neighbor.





Fact

Every tree on at least two vertices has at least two leaves.

Fact

If v is a leaf of a tree T, then $T \setminus v$ is a tree.

Definition

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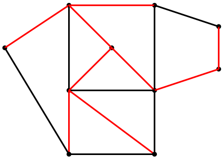

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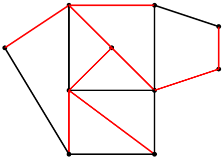

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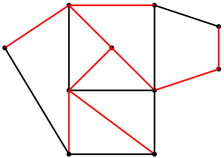

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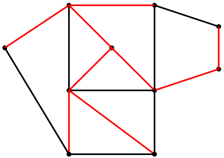

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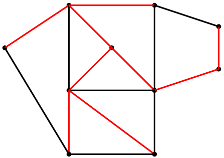

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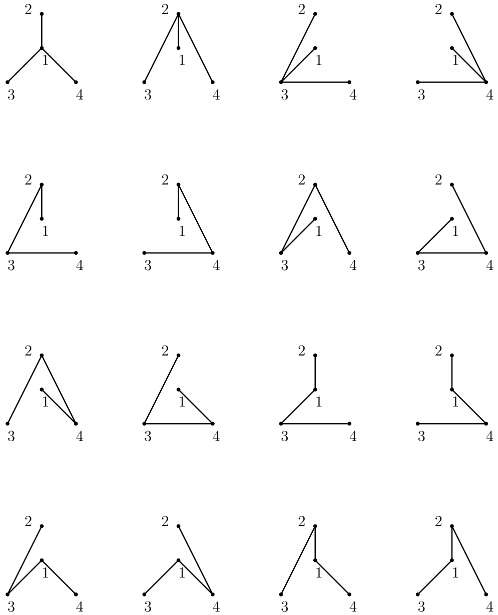
- For $n = 2$, there is one such tree.
- For $n = 3$, there are three such trees.
- For $n = 4$, there are 16 such trees.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Lemma 2.4

Let $n \geq 2$ be an integer, and let $S \subseteq \mathbb{N}$ be such that $|S| = n$. Then the number of trees on the vertex set S is n^{n-2}.

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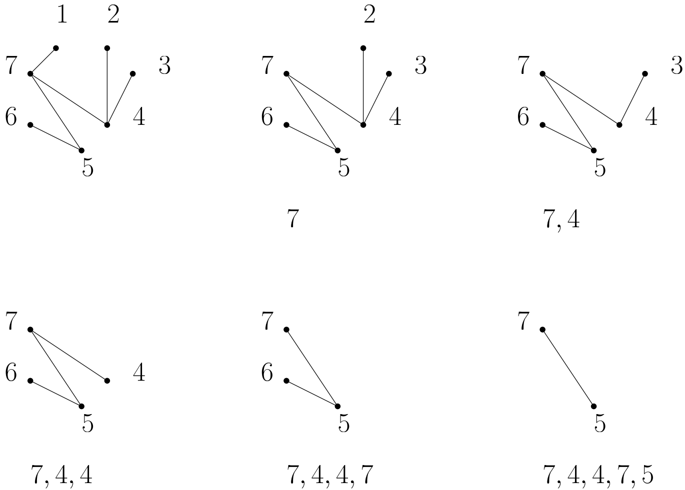
Definition

We define the *Prüfer code* of integer trees on at least two vertices recursively, as follows:

- for any integer tree T on exactly two vertices, the Prüfer code of T, denoted by $P(T)$, is the empty sequence;
- for any integer tree T on at least three vertices, we define the Prüfer code of T to be $P(T) := a_i, P(T \setminus i)$, where i is the smallest leaf of T, and a_i is the unique neighbor of i in T.[a]

[a] So, $P(T)$ is obtained by adding a_i to the front of $P(T \setminus i)$.

- For example, the Prüfer code of the tree in the top left corner is 7, 4, 4, 7, 5, as shown below:



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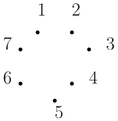
- For an integer $n \geq 2$, an n-element set $S \subseteq \mathbb{N}$, and an $(n-2)$-term sequence P, with terms in S, we proceed as follows.
  1. If $n \geq 3$, then we let i be the smallest element of S that is not in P, and we let a_i be the first term of P. We make i and a_i adjacent, we delete i from S, and we delete the first term of P.
  2. We repeat the process until S only has two elements left, and P is the empty sequence. At this point, we make the last two remaining elements of S adjacent.

- For example, the tree on the vertex set $S = \{1, 2, 3, 4, 5, 6, 7\}$ whose Prüfer code is $7, 4, 4, 7, 5$ is the tree on the bottom of the picture (e is the empty sequence).



$P = 7, 4, 4, 7, 5$
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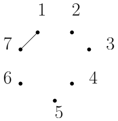

$P = 4, 4, 7, 5$
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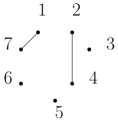

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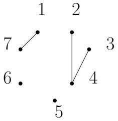

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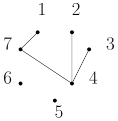
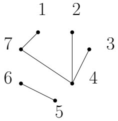


$P = 5$
$S = \{5, 6, 7\}$



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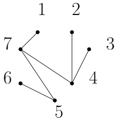

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Cayley's formula

For all $n \geq 2$, the number of spanning trees of K_n is n^{n-2}.

Proof. This follows immediately from Lemma 2.4, for $S = \{1, \ldots, n\}$.

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- One proof uses the "Laplacians" (matrices).
- In fact, one can use the "Laplacian" of an arbitrary graph (on vertex set $\{1, \ldots, n\}$) to compute the number of spanning trees of that graph.
- We give the formula without proof.

Definition

Suppose that $n \geq 2$ is an integer, and that G is a graph on the vertex set $\{1, \dots, n\}$. Then the *Laplacian* of G is the matrix $Q = [q_{i,j}]_{n \times n}$ given by

$$q_{i,j} = \begin{cases} d_G(i) & \text{if} \quad i = j \\ -1 & \text{if} \quad i \neq j \text{ and } ij \in E(G) \\ 0 & \text{if} \quad i \neq j \text{ and } ij \notin E(G) \end{cases}$$

Theorem 2.5

Let $n \geq 2$ be an integer, let G be any graph on the vertex set $\{1, \dots, n\}$, and let Q be the Laplacian of G. Then the number of spanning trees of G is precisely $\det(Q_{1,1})$.[a]

[a] $Q_{1,1}$ is the matrix obtained from Q by deleting the first row and first column.

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Solution. Fix an integer $n \geq 2$, and consider the complete graph on the vertex set $\{1, \ldots, n\}$. Then the Laplacian of this graph is the $n \times n$ matrix

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The matrix $Q_{1,1}$ has exactly the same form, only it is of size $(n-1) \times (n-1)$. Since $\det(Q_{1,1}) = n^{n-2}$ (details: Lecture Notes), Theorem 2.5 guarantees that the number of spanning trees of K_n is n^{n-2}. This proves Cayley's formula.