NDMI011: Combinatorics and Graph Theory 1

Lecture #8

Graph connectivity and Menger's theorems

Irena Penev

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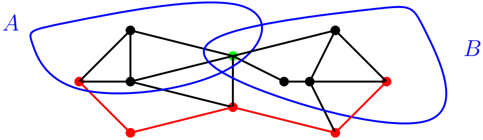
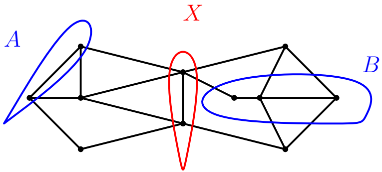
Definition

For a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, an *A-B path* in G, or a *path from A to B* in G, is either a one-vertex path whose sole vertex is in $A \cap B$, or a path on at least two vertices whose one endpoint is in A and whose other endpoint is in B.



Definition

Given a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, we say that a set $X \subseteq V(G)$ *separates* A from B in G if every path from A to B in G contains at least one vertex of X. Note that this implies that $A \cap B \subseteq X$.



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Given a graph G and a non-negative integer k, we say that G is *k-vertex-connected*, or simply *k-connected*, if $|V(G)| \geq k+1$ and for all $X \subseteq V(G)$ such that $|X| \leq k-1$, we have that $G \setminus X$ is connected.

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- If $k = \kappa(G)$, then either $G = K_{k+1}$ or there exists a set of k vertices whose deletion from G yields a disconnected graph.
- If there exists a set of at most k vertices whose deletion from G yields a disconnected graph, then $\kappa(G) \leq k$.

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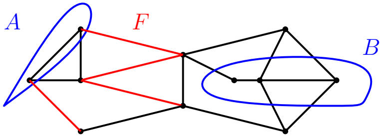
Given a graph G and disjoint sets $A, B \subseteq V(G)$, we say that a set $F \subseteq E(G)$ *separates* A from B in G if every path from A to B contains at least one edge of F.



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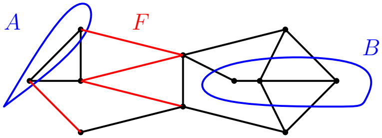

Definition

Given a graph G and a non-negative integer ℓ, we say that G is *ℓ-edge-connected* if $|V(G)| \geq 2$ and for all $F \subseteq E(G)$ such that $|F| \leq \ell - 1$, we have that $G \setminus F$ is connected.

Definition

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- If there exists a set of at most ℓ edges whose deletion from G yields a disconnected graph, then $\lambda(G) \leq \ell$.

Proposition 1.1

Let G be a graph on at least two vertices. Then

(a) for all edges $e \in E(G)$, $\kappa(G) - 1 \leq \kappa(G \setminus e) \leq \kappa(G)$;

(b) for all sets $F \subseteq E(G)$, $\kappa(G \setminus F) \leq \kappa(G)$.

Proposition 1.2

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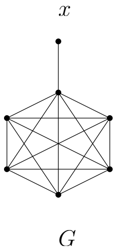
- For instance, for the graph G represented below, we have that $\kappa(G) = \lambda(G) = 1$, but $\kappa(G \setminus x) = \lambda(G \setminus x) = 5$.



Theorem 1.3

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It remains to show that $\kappa(G) \leq \lambda(G)$. Fix a set $F \subseteq E(G)$ such that $|F| = \lambda(G)$ and $G \setminus F$ is disconnected.

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It remains to show that $\kappa(G) \leq \lambda(G)$. Fix a set $F \subseteq E(G)$ such that $|F| = \lambda(G)$ and $G \setminus F$ is disconnected.

> ***Claim.*** *If C is the vertex set of a component of $G \setminus F$, then no edge of F has both its endpoints in C.*

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> ***Claim.*** *If C is the vertex set of a component of $G \setminus F$, then no edge of F has both its endpoints in C.*

Proof of the Claim. Suppose some edge $e \in F$ be an edge that has both its endpoints in C. Then $G \setminus (F \setminus \{e\})$ is still disconnected, contrary to the fact that $|F \setminus \{e\}| = |F| - 1 = \lambda(G) - 1$. This proves the Claim.

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Let G be a graph on at least two vertices. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proof (continued). Reminder: $F \subseteq E(G)$, $|F| = \lambda(G)$, $G \setminus F$ is disconnected. WTS $\kappa(G) \leq \lambda(G)$.

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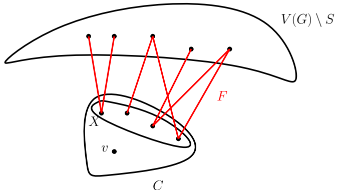
Suppose first that there exists a vertex $v \in V(G)$ that is not incident with any edge in F. Let C be the vertex set of the component of $G \setminus F$ that contains v. By the Claim, no edge in F has both endpoints in C. Now, let X be the set of all vertices in C that are incident with an edge in F. Then $|X| \leq |F| = \lambda(G)$ and $G \setminus X$ is disconnected. So, $\kappa(G) \leq \lambda(G)$.



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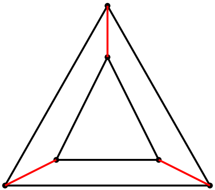
It remains to consider the case when every vertex of G is incident with an edge of F.



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Proof (continued). Reminder: $F \subseteq E(G)$, $|F| = \lambda(G)$, $G \setminus F$ is disconnected, and every vertex in C is incident with an edge of F. WTS $\kappa(G) \leq \lambda(G)$.

Fix any $v \in V(G)$; we claim that $d_G(v) \leq \lambda(G)$. Let C be the vertex set of the component of $G \setminus F$ that contains v. Then for all distinct $u, w \in N_C(v)$, we have (by the Claim) that $uw \notin F$, and so (since every vertex of G is incident with an edge in F) u and w are incident with distinct edges of F. This implies that $d_G(v) \leq |F| = \lambda(G)$.

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Proof (continued). Reminder: $F \subseteq E(G)$, $|F| = \lambda(G)$, $G \setminus F$ is disconnected, and every vertex in C is incident with an edge of F. WTS $\kappa(G) \leq \lambda(G)$.

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Proof (continued). Reminder: $F \subseteq E(G)$, $|F| = \lambda(G)$, $G \setminus F$ is disconnected, $\lambda(G) = \Delta(G)$. WTS $\kappa(G) \leq \lambda(G)$.

Now, if G is a complete graph, then $|V(G)| = \Delta(G) + 1$, and we see that $\kappa(G) = \Delta(G) = \lambda(G)$. So assume that G is not complete, and fix some $x \in V(G)$ that has a non-neighbor in G. Then $G \setminus N_G(x)$ is disconnected, and we have that $|N_G(x)| = d_G(x) \leq \Delta(G) = \lambda(G)$. So, $\kappa(G) \leq \lambda(G)$.

Definition

A *vertex-cutset* of a graph G is any set $X \subsetneq V(G)$ such that $G \setminus X$ has more components than G. Similarly, an *edge-cutset* of G is any set $F \subseteq E(G)$ such that $G \setminus F$ has more components than G.

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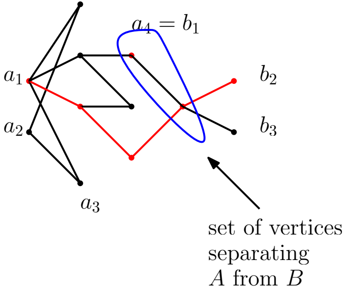
- By definition, no graph G has a vertex-cutset of size strictly smaller than $\kappa(G)$.
- Similarly, no graph G has an edge-cutset of size strictly smaller than $\lambda(G)$.

Menger's theorem (vertex version)

Let G be a graph, and let $A, B \subseteq V(G)$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of pairwise disjoint A-B paths in G.



$A = \{a_1, a_2, a_3, a_4\}$

$B = \{b_1, b_2, b_3\}$

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Proof (outline).

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Proof (outline). Assume inductively that the theorem is true for graphs on fewer than $|E(G)|$ edges. Let k be the minimum number of vertices separating A from B in G.

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(i) is "obvious." Let's prove (ii). If $E(G) = \emptyset$, then $|A \cap B| = k$, and there are k pairwise disjoint A-B paths in G. So assume that G has at least one edge, say xy.

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G G/xy

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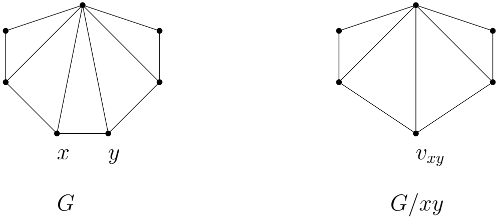

If x or y belongs to A, then let $A' = (A \setminus \{x, y\}) \cup \{v_{xy}\}$, and otherwise, let $A' = A$. Similarly, if x or y belongs to B, then let $B' = (B \setminus \{x, y\}) \cup \{v_{xy}\}$, and otherwise, let $B' = B$.

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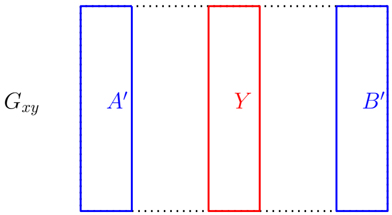
Proof (outline, continued). Let $Y \subseteq V(G_{xy})$ be a minimum-sized set of vertices separating A' from B' in G_{xy}. By the induction hypothesis, there are $|Y|$ many pairwise disjoint paths in G_{xy} from A' to B', and it readily follows that there are at least $|Y|$ many pairwise disjoint paths in G from A to B. So, if $|Y| \geq k$, then we are done.



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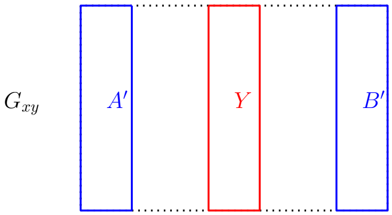
Proof (outline, continued). From now on, we assume that $|Y| \leq k-1$. Then $v_{xy} \in Y$, for otherwise, Y would separate A from B in G, contrary to the fact that $|Y| \leq k-1$. Now $X := (Y \setminus \{v_{xy}\}) \cup \{x, y\}$ separates A from B in G, and we have that $|X| = |Y| + 1$. Note that this implies that $|X| = k$. Set $X = \{x_1, \ldots, x_k\}$.



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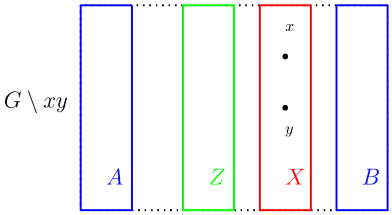
Proof (outline, continued). We now consider the graph $G \setminus xy$. Since $x, y \in X$, we know that any set of vertices separating A from X in $G \setminus xy$ also separates A from B in G; consequently, any such set has at least k vertices.



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Proof (outline, continued). So, by the induction hypothesis, there are k pairwise disjoint paths from A to X in G, call them $P_1, \ldots, P_k$.

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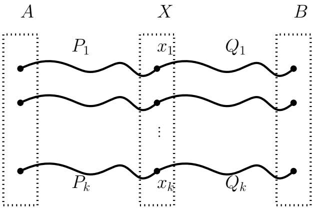
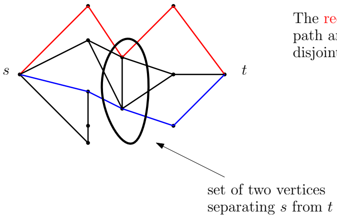
Proof (outline, continued). So, by the induction hypothesis, there are k pairwise disjoint paths from A to X in G, call them $P_1, \ldots, P_k$. Similarly, there are k pairwise disjoint paths from B to X in G, call them $Q_1, \ldots, Q_k$.



Corollary 2.1

Let G be a graph, and let $s, t \in V(G)$ be distinct, non-adjacent vertices of G. Then the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G.



The red and blue path are internally disjoint.

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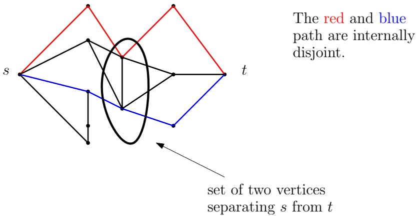
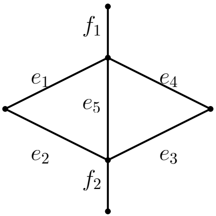


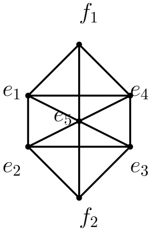
Proof (outline). Apply Menger's theorem (vertex version) to the graph $G \setminus \{s, t\}$ and sets $S = N_G(s)$ and $T = N_G(t)$.

Definition

The *line graph* of a graph G, denoted by $L(G)$, is the graph whose vertex set is $E(G)$, and in which $e, f \in L(V(G)) = E(G)$ are adjacent if and only if e and f share an endpoint in G.



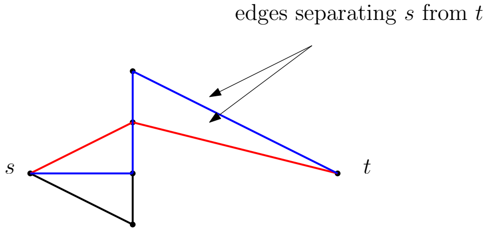
G



$L(G)$

Menger's theorem (edge version)

Let G be a graph, and let $s, t \in V(G)$ be distinct vertices of G. Then the minimum number of edges separating s from t in G is equal to the maximum number of pairwise edge-disjoint s-t paths in G.



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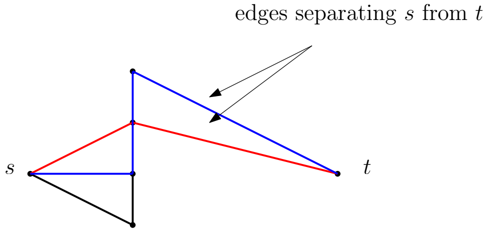

Proof (outline). Apply Menger's theorem (vertex version) to the graph $L(G)$ and the sets $S = \{e \in E(G) \mid e \text{ is incident with } s\}$ and $T = \{e \in E(G) \mid e \text{ is incident with } t\}$.

The global version of Menger's theorem

Let G be a graph on at least two vertices, and let $k, \ell \geq 0$ be integers.

(a) G is k-connected if and only if for all distinct $s, t \in V(G)$, there are k pairwise internally disjoint s-t paths in G.

(b) G is ℓ-edge-connected if and only if for all distinct $s, t \in E(G)$, there are ℓ pairwise edge-disjoint s-t paths in G.

Proof. HW.