NDMI011: Combinatorics and Graph Theory 1

Lecture #8
Graph connectivity and Menger's theorems

Irena Penev

In what follows, all graphs are finite, simple (i.e. have no loops and no parallel edges), and non-null.

1 Vertex and edge connectivity

For a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, an *A-B path* in G, or a *path from A to B* in G, is either a one-vertex path whose sole vertex is in $A \cap B$, or a path on at least two vertices whose one endpoint is in A and whose other endpoint is in B.

Given a graph G and (not necessarily disjoint) sets $A, B \subseteq V(G)$, we say that a set $X \subseteq V(G)$ *separates* A from B in G if every path from A to B in G contains at least one vertex of X. Note that this implies that $A \cap B \subseteq X$.[1]

Given a graph G and a non-negative integer k, we say that G is *k-vertex-connected*, or simply *k-connected*, if $|V(G)| \geq k+1$ and for all $X \subseteq V(G)$ such that $|X| \leq k-1$, we have that $G \setminus X$ is connected. Note that this means that every (non-null) graph is 0-connected, and that every connected graph on at least two vertices is 1-connected.[2] The *connectivity* of a graph G, denoted $\kappa(G)$, is the largest integer k such that G is k-connected. Note that if $k = \kappa(G)$, then either $G = K_{k+1}$ or there exists a set of k vertices whose deletion from G yields a disconnected graph. Furthermore, if there exists a set of at most k vertices whose deletion from G yields a disconnected graph, then $\kappa(G) \leq k$.

Given a graph G and disjoint sets $A, B \subseteq V(G)$, we say that a set $F \subseteq E(G)$ *separates* A from B in G if every path from A to B contains at least one edge of F.

Given a graph G and a non-negative integer ℓ, we say that G is *ℓ-edge-connected* if $|V(G)| \geq 2$ and for all $F \subseteq E(G)$ such that $|F| \leq \ell - 1$, we

[1] Indeed, if $x \in A \cap B$, then x counts as a one-vertex path from A to B. So, any set of vertices that separates A from B must include $A \cap B$ as a subset.

[2] However, K_1 is **not** 1-connected.

have that $G \setminus F$ is connected. The *edge-connectivity* of a graph G on at least two vertices, denoted by $\lambda(G)$, is the largest integer ℓ such that G is ℓ-edge-connected. Note that if $\ell = \lambda(G)$, then there exists a set of ℓ edges whose deletion from G yields a disconnected graph. Furthermore, if there exists a set of at most ℓ edges whose deletion from G yields a disconnected graph, then $\lambda(G) \leq \ell$.

Proposition 1.1. *Let G be a graph on at least two vertices. Then*

(a) for all edges $e \in E(G)$, $\kappa(G) - 1 \leq \kappa(G \setminus e) \leq \kappa(G)$;

(b) for all sets $F \subseteq E(G)$, $\kappa(G \setminus F) \leq \kappa(G)$.

Proof. Clearly, (b) follows from (a) by an easy induction. It remains to prove (a). Fix $e \in E(G)$.

We first show that $\kappa(G \setminus e) \geq \kappa(G) - 1$. Since G is $\kappa(G)$-connected, we know that G (and consequently, $G \setminus e$ as well) has at least $\kappa(G) + 1$ vertices. Now, fix $X \subseteq V(G)$ such that $|X| \leq \kappa(G) - 2$; we must show that $(G \setminus e) \setminus X$ is connected. Suppose first that e is incident with some vertex in X. Then $(G \setminus e) \setminus X = G \setminus X$. Since $|X| \leq \kappa(G) - 2$, we see that $G \setminus X$ is connected, and it follows that $(G \setminus e) \setminus X$ is connected. It remains to consider the case when e is not incident with any vertex in X. Set $e = x_1x_2$ (i.e. let x_1 and x_2 be the endpoints of e). Set $X_1 = X \cup \{x_1\}$ and $X_2 \cup \{x_2\}$. Then $|X_1| = |X_2| = \kappa(G) - 1$, and we deduce that $G \setminus X_1$ and $G \setminus X_2$ are connected. Now, since $x_2 \in V(G) \setminus X_1$, and since $G \setminus X_1$ is a connected graph on at least two vertices, we see that x_2 is adjacent to some vertex in $u \in V(G) \setminus X_1$; since $x_1 \in X_1$, we see that $u \neq x_1$. Now, $(G \setminus e) \setminus X$ can be obtained from the connected graph $G \setminus X_2$ by adding to it the vertex x_2 and making it adjacent to all vertices in $N_G(x_2) \setminus \{x_1\}$. Since $u \in N_G(x_2) \setminus \{x_1\}$, we see that x_2 is not an isolated vertex of $(G \setminus e) \setminus X$, and we deduce that $(G \setminus e) \setminus X$ is connected. This proves that $\kappa(G \setminus e) \leq \kappa(G) - 1$.

It remains to show that $\kappa(G \setminus e) \leq \kappa(G)$. By definition, $|V(G)| \geq \kappa(G)+1$. If G has precisely $\kappa(G) + 1$ vertices, then so does $G \setminus e$, and it follows from the definition that $\kappa(G \setminus e) \leq \kappa(G)$. It remains to consider the case when $|V(G)| \geq \kappa(G) + 2$. In this case, there exists a set $X \subsetneq V(G)$ of size $\kappa(G)$ such that $G \setminus X$ is disconnected. But then $(G \setminus e) \setminus X$ is disconnected as well, and it follows that $\kappa(G \setminus e) \leq \kappa(G)$. □

Proposition 1.2. *Let G be a graph on at least two vertices. Then*

(a) for all edges $e \in E(G)$, $\lambda(G) - 1 \leq \lambda(G \setminus e) \leq \lambda(G)$;

(b) for all sets $F \subseteq E(G)$, $\lambda(G \setminus F) \leq \lambda(G)$.

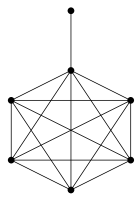
Proof. Clearly, (b) follows from (a) by an easy induction. It remains to prove (a). Fix $e \in E(G)$.

We first show that $\lambda(G \setminus e) \geq \lambda(G) - 1$. Fix $F \subseteq E(G \setminus e)$ such that $|F| \leq \lambda(G) - 2$. Set $F' = F \cup \{e\}$; then $|F'| \leq \lambda(G) - 1$, and we deduce that $G \setminus F'$ is connected. But $(G \setminus e) \setminus F = G \setminus F'$, and we deduce that $(G \setminus e) \setminus F$ is connected. This proves that $\lambda(G \setminus e) \geq \lambda(G) - 1$.

It remains to show that $\lambda(G \setminus e) \leq \lambda(G)$. Fix $F \subseteq E(G)$ with $|F| = \lambda(G)$, such that $G \setminus F$ is disconnected. Set $F' = F \setminus \{e\}$; then $|F'| \leq \lambda(G)$. Furthermore, we have that $(G \setminus e) \setminus F' = G \setminus F$, and we deduce that $(G \setminus e) \setminus F'$ is disconnected. Since $|F'| \leq \lambda(G)$, we see that $\lambda(G \setminus e) \leq \lambda(G)$. □

We note that, unlike edge deletion, vertex deletion sometimes increases connectivity. For instance, for the graph G represented below, we have that $\kappa(G) = \lambda(G) = 1$, but $\kappa(G \setminus x) = \lambda(G \setminus x) = 5$.

x



G

Recall that for a graph G, $\delta(G)$ is the minimum and $\Delta(G)$ the maximum degree in G, i.e. $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ and $\Delta(G) = \max\{d_G(v) \mid v \in V(G)\}$.

Theorem 1.3. *Let G be a graph on at least two vertices. Then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Proof. We first prove that $\lambda(G) \leq \delta(G)$. Fix a vertex $v \in V(G)$ such that $d_G(v) = \delta(G)$, and let F be the set of all edges of G that are incident with v. Clearly, $G \setminus F$ is disconnected, and it follows that $\lambda(G) \leq \delta(G)$.

It remains to show that $\kappa(G) \leq \lambda(G)$. Fix a set $F \subseteq E(G)$ such that $|F| = \lambda(G)$ and $G \setminus F$ is disconnected.

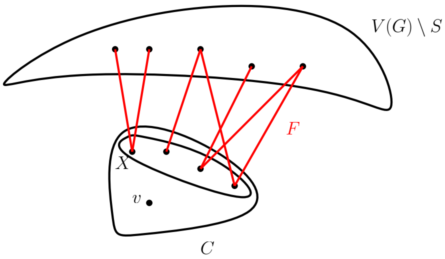
> **Claim.** If C is the vertex set of a component of $G \setminus F$, then no edge of F has both its endpoints in C.

Proof of the Claim. Suppose otherwise. Let C be the vertex set of a component of $G \setminus F$,[3] and let $e \in F$ be an edge that has both its endpoints in C. Then $G \setminus (F \setminus \{e\})$ is still disconnected,[4] contrary to the fact that $|F \setminus \{e\}| = |F| - 1 = \lambda(G) - 1$. This proves the Claim. ■

[3] Since $G \setminus F$ is disconnected, this implies that C and $V(G) \setminus C$ are both non-empty, and there are no edges between them.

[4] This is because there are still no edges between C and $V(G) \setminus C$, and both C and $V(G) \setminus C$ are non-empty.

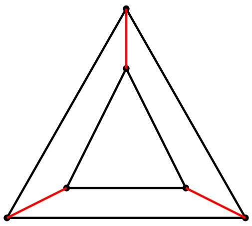
Suppose first that there exists a vertex $v \in V(G)$ that is not incident with any edge in F. Let C be the vertex set of the component of $G \setminus F$ that contains v. By the Claim, no edge in F has both endpoints in C. Now, let X be the set of all vertices in C that are incident with an edge in F. Then $|X| \leq |F| = \lambda(G)$ and $G \setminus X$ is disconnected. So, $\kappa(G) \leq \lambda(G)$.



It remains to consider the case when every vertex of G is incident with an edge of F.[5] Fix any $v \in V(G)$; we claim that $d_G(v) \leq \lambda(G)$. Let C be the vertex set of the component of $G \setminus F$ that contains v. Then for all distinct $u, w \in N_C(v)$, we have (by the Claim) that $uw \notin F$, and so (since every vertex of G is incident with an edge in F) u and w are incident with distinct edges of F. This implies that $d_G(v) \leq |F| = \lambda(G)$.[6] Since we chose v arbitrarily, this implies that $\Delta(G) \leq \lambda(G)$; we already saw that $\lambda(G) \leq \delta(G)$, and we now deduce that $\lambda(G) = \Delta(G)$. Now, if G is a complete graph, then $|V(G)| = \Delta(G) + 1$, and we see that $\kappa(G) = \Delta(G) = \lambda(G)$. So assume that G is not complete, and fix some $x \in V(G)$ that has a non-neighbor in G. Then $G \setminus N_G(x)$ is disconnected, and we have that $|N_G(x)| = d_G(x) \leq \Delta(G) = \lambda(G)$. So, $\kappa(G) \leq \lambda(G)$. □

Terminology: A *vertex-cutset* of a graph G is any set $X \subsetneqq V(G)$ such that $G \setminus X$ has more components than G.[7] Similarly, an *edge-cutset* of G is any set $F \subseteq E(G)$ such that $G \setminus F$ has more components than G.

[5]For an example, see the graph below, with the edges of F in red.



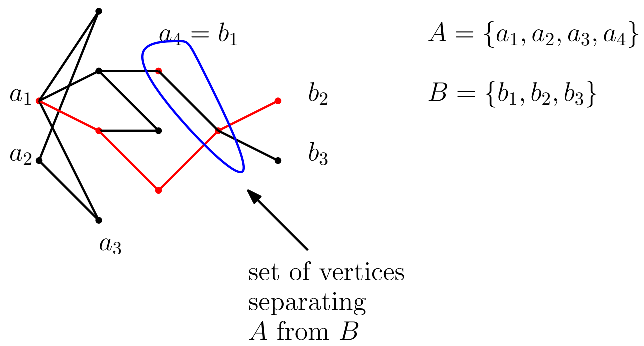
[6]Let us explain this in more detail. Let F_1 be the set of all edges in F that are incident with v. Then $d_G(v) = |F_1| + |N_C(v)|$. Further, by what we just showed, every vertex in $N_C(v)$ is incident with an edge of F, and no two vertices in $N_C(v)$ are incident with the same edge of F. It is also clear that no vertex in $N_C(v)$ is incident with an edge of F_1. So, $|N_C(v)| \leq |F \setminus F_1|$, and we deduce that $d_G(v) \leq |F_1| + |F \setminus F_1| = |F| = \lambda(G)$.

[7]So, if G is connected, then a vertex-cutset of G is any set $X \subsetneqq V(G)$ such that $G \setminus X$ is disconnected.

By definition, no graph G has a vertex-cutset of size strictly smaller than $\kappa(G)$. Similarly, no graph G has an edge-cutset of size strictly smaller than $\lambda(G)$.

2 Menger's theorems

Menger's theorem (vertex version). *Let G be a graph, and let $A, B \subseteq V(G)$.*[8] *Then the minimum number of vertices separating A from B in G is equal to the maximum number of pairwise disjoint A-B paths in G.*[9]



Proof. We assume inductively that the theorem holds for graphs that have fewer than $|E(G)|$ edges. More precisely, we assume that for all graphs G' such that $|E(G')| < |E(G)|$, and all sets $A', B' \subseteq V(G')$, the minimum number of vertices separating A' from B' in G' is equal to the maximum number of pairwise disjoint A'-B' paths in G'. We must prove that this holds for G as well. From now on, we let k be the minimum number of vertices separating A from B in G.

First, we claim that there can be no more than k pairwise disjoint paths from A to B in G. Indeed, let $X \subseteq V(G)$ be a k-vertex set separating A from B in G, and let $\mathcal{P}$ be any collection of pairwise disjoint paths from A to B. By definition, every path in $\mathcal{P}$ contains at least one vertex of X, and since paths in $\mathcal{P}$ are pairwise disjoint, no two paths in $\mathcal{P}$ contain the same vertex of X. So, $|\mathcal{P}| \leq |X| = k$, as we had claimed.

It remains to show that there are at least k pairwise disjoint paths from A to B. Clearly, for any set $X \subseteq V(G)$ separating A from B in G, we have that $A \cap B \subseteq X$; consequently, $|A \cap B| \leq k$. Now, if $E(G) = \emptyset$, then $A \cap B$ separates A from B in G, and so $|A \cap B| = k$; in this case, the vertices of $A \cap B$ form k pairwise disjoint one-vertex paths from A to B, and we are done. From now on, we assume that G has at least one edge, say xy. Let

[8] A and B need not be disjoint.

[9] "Pairwise disjoint" here means that no two paths have a vertex in common (and consequently, no two paths have an edge in common).

$G_{xy} := G/xy$, i.e. let G_{xy} be the graph obtained from G by contracting the edge xy, and let v_{xy} be the vertex obtained by contracting xy.[10]

G $\qquad\qquad$ G/xy

Now, if x or y belongs to A, then let $A' = (A \setminus \{x, y\}) \cup \{v_{xy}\}$, and otherwise, let $A' = A$. Similarly, if x or y belongs to B, then let $B' = (B \setminus \{x, y\}) \cup \{v_{xy}\}$, and otherwise, let $B' = B$.

Let $Y \subseteq V(G_{xy})$ be a minimum-sized set of vertices separating A' from B' in G_{xy}.[11] By the induction hypothesis, there are $|Y|$ many pairwise disjoint paths in G_{xy} from A' to B', and it readily follows[12] that there are at least $|Y|$ many pairwise disjoint paths in G from A to B. So, if $|Y| \geq k$,[13] then we are done. From now on, we assume that $|Y| \leq k - 1$. Then $v_{xy} \in Y$, for otherwise, Y would separate A from B in G,[14] contrary to the fact that $|Y| \leq k - 1$. Now $X := (Y \setminus \{v_{xy}\}) \cup \{x, y\}$ separates A from B in G,[15] and we have that $|X| = |Y| + 1$. Note that this implies that $|X| = k$.[16] Set $X = \{x_1, \dots, x_k\}$.

We now consider the graph $G \setminus xy$, i.e. the graph obtained from G by deleting the edge xy.[17] Since $x, y \in X$, we know that any set of vertices separating A from X in $G \setminus xy$ also separates A from B in G;[18] consequently, any such set has at least k vertices, and so by the induction hypothesis,

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[10] Formally, we have that v_{xy} is some vertex that does not belong to $V(G)$, and that G_{xy} is the graph with vertex set $V(G_{xy}) = \big(V(G) \setminus \{x, y\}\big) \cup \{v_{xy}\}$ and edge set $E(G_{xy}) = \{e \in E(G) \mid e \text{ is incident neither with } x \text{ nor with } y \text{ in } G\} \cup \{vv_{xy} \mid v \in V(G), v \text{ is incident with } x \text{ or } y \text{ in } G\}$.

[11] This means that for all sets $Y' \subseteq V(G_{xy})$ separating A from B in G_{xy}, we have that $|Y| \leq |Y'|$.

[12] Details?

[13] In fact, it is not possible that $|Y| > k$ (details?), but we do not need this stronger fact.

[14] Proof?

[15] Proof?

[16] Indeed, since $|Y| \leq k - 1$, we have that $|X| \leq k$. On the other hand, since X separates A from B in G, we know that $|X| \geq k$. So, $|X| = k$.

[17] So, $V(G \setminus xy) = V(G)$ and $E(G \setminus xy) = E(G) \setminus \{xy\}$.

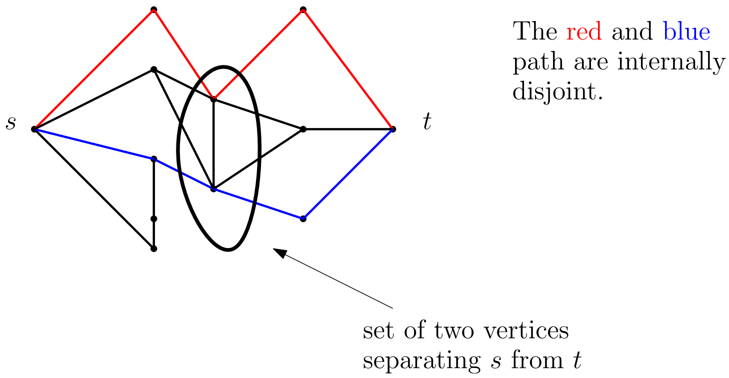
[18] Let us check this. Let Z be any set of vertices separating A from X in $G \setminus xy$, and let $p_1, \dots, p_t$, with $p_1 \in A$ and $p_t \in B$, be a path from A to B in G. Then some vertex of $p_1, \dots, p_t$ belongs to X; let $i \in \{1, \dots, t\}$ be the smallest index such that $p_i \in X$. Then $p_1, \dots, p_i$ is a path from A to X in G. Furthermore, since $p_1, \dots, p_i$ contains exactly one vertex of X, and since $x, y \in X$, we see that the path $p_1, \dots, p_i$ does not use the edge xy; consequently, $p_1, \dots, p_i$ is a path from A to X in $G \setminus xy$, and we deduce that this path (and consequently, the path $p_1, \dots, p_t$ as well) contains a vertex of Z.

there are k pairwise disjoint paths from A to X in G, call them $P_1, \ldots, P_k$. Similarly, there are k pairwise disjoint paths from B to X in G, call them $Q_1, \ldots, Q_k$. We may assume that for all $i \in \{1, \ldots, k\}$, x_i is an endpoint both of P_i and of Q_i. So, $P_1 - x_1 - Q_1, \ldots, P_k - x_k - Q_k$ are walks from A to B. But in fact, each of these walks is a path, for otherwise, it would contain a path from A to B that contains no vertex of X.[19] So, there are at least k paths from A to B in G. □

Given a graph G and distinct vertices $s, t \in V(G)$, two paths from s to t in G are *internally disjoint* if they have no vertices in common except the endpoints s and t.

The following corollary is also often referred to as the vertex version of Menger's theorem.

Corollary 2.1. *Let G be a graph, and let $s, t \in V(G)$ be distinct, non-adjacent vertices of G. Then the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G.*

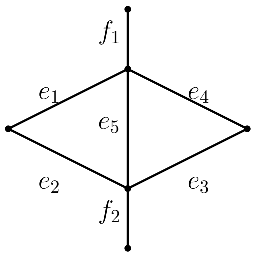


Proof. Let $S = N_G(s)$ and $T = N_G(t)$. Obviously, the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating S from T in $G \setminus \{s, t\}$.[20] Similarly, the maximum number of pairwise internally disjoint s-t paths in G is equal to the maximum number of pairwise disjoint S-T paths in G. By Menger's theorem (vertex version), the minimum number of vertices separating S from T in $G \setminus \{s, t\}$ is equal to the maximum number of pairwise disjoint S-T paths in $G \setminus \{s, t\}$. So, the minimum number of vertices of $V(G) \setminus \{s, t\}$ separating s from t in G is equal to the maximum number of pairwise internally disjoint s-t paths in G. This completes the argument. □

[19] Details?

[20] Indeed, for any set $X \subseteq V(G) \setminus \{s, t\}$, we have that X separates s from t in G if and only if X separates S from T in $G \setminus \{s, t\}$.

Our next goal is to prove the edge version of Menger's theorem. The *line graph* of a graph G, denoted by $L(G)$, is the graph whose vertex set is $E(G)$, and in which $e, f \in L(V(G)) = E(G)$ are adjacent if and only if e and f share an endpoint in G.



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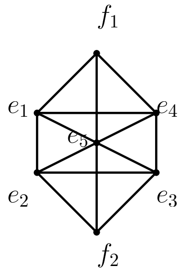


$L(G)$

Proposition 2.2. *Let G be a graph, let $s, t \in V(G)$ be distinct vertices of G, let S be the set of all edges in G incident with s, and let T be the set of all edges in G incident with t. Let $X \subseteq E(G)$. Then X separates s from t in G if and only if X separates S from T in $L(G)$.*

Proof. Suppose that X separates s from t in G; we must show that X separates S from T in G. Suppose otherwise. Then there exists some path $e_1, \dots, e_r$ in $L(G)$ that does not contain any vertex (in $L(G)$) from X.[21] For each $i \in \{1, \dots, r-1\}$, let v_i be a common vertex of e_i and e_{i+1}.[22] Then $s, v_1, \dots, v_{r-1}, t$ is a walk in $L(G)$ from s to t that uses only edges $e_1, \dots, e_r$, and consequently, does not use any edge of X. It follows that there is a path from s to t in G that does not use any edges of X, contrary to the fact that X separates s from t in G. This proves that X indeed separates S from T in G.

Suppose now that X does not separate s from t in G; we must show that X does not separate S from T in $L(G)$. Since X does not separate s from t in G, we know that there is a path $v_1, \dots, v_r$ in G, with $v_1 = s$ and $v_r = t$, that does not use any edge of X. But now $v_1v_2, v_2v_3, \dots, v_{r-1}v_r$ is a path from S to T in $L(G)$ that does not use any vertex (in $L(G)$) in X. So, X does not separate S from T in $L(G)$. $\square$

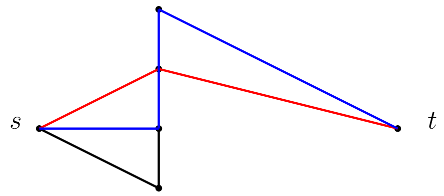
Proposition 2.3. *Let G be a graph, let $s, t \in V(G)$ be distinct vertices of G, let S be the set of all edges in G incident with s, and let T be the set of all edges in G incident with t. Let ℓ be a non-negative integer. Then the following are equivalent:*

(i) there are ℓ pairwise edge-disjoint s-t paths in G;

[21] Note that $e_1, \dots, e_r$ are vertices of $L(G)$, and consequently, edges of G.

[22] Such a vertex exists because e_i and e_{i+1} are adjacent vertices of $L(G)$, and consequently, they are edges of G that share an endpoint.

(ii) there are ℓ pairwise disjoint S-G paths in $L(G)$.

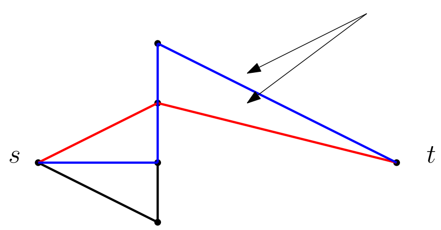


The red and blue path are edge-disjoint.

Proof. Suppose first that (i) holds, and fix ℓ pairwise edge-disjoint s-t paths in G, say $P_1, \dots, P_\ell$. For all $i \in \{1, \dots, \ell\}$, set $P_i = v_1^i, \dots, v_{r_i}^i$. Now, for all $i \in \{1, \dots, \ell\}$, set $P_i^L = v_1^i v_2^i, v_2^i v_3^i, \dots, v_{r_i-1}^i v_{r_i}^i$ (with $v_1^i = s$ and $v_{r_i}^i = t$). Clearly, $P_1^L, \dots, P_\ell^L$ are pairwise disjoint S-T paths in $L(G)$.

Suppose now that (ii) holds, and fix ℓ pairwise disjoint S-T paths in G, say $Q_1^L, \dots, Q_\ell^L$. For all $i \in \{1, \dots, \ell\}$, set $Q_i^L = e_1^i, \dots, e_{r_i}^i$. Now, for all $i \in \{1, \dots, \ell\}$ and $j \in \{1, \dots, r_i\}$, let v_j^i be a common vertex of the edges e_j^i and e_{j+1}^i in G, and set $Q_i = s, v_1^i, \dots, v_{r_i-1}^i, t$. Then $Q_1, \dots, Q_\ell$ are pairwise edge-disjoint s-t walks in G, and we deduce that there are ℓ pairwise edge-disjoint s-t paths in G. $\square$

Menger's theorem (edge version). *Let G be a graph, and let $s, t \in V(G)$ be distinct vertices of G. Then the minimum number of edges separating s from t in G is equal to the maximum number of pairwise edge-disjoint s-t paths in G.*



Proof. Let S be the set of all edges in G incident with s, and let T be the set of all edges in G incident with t. By Proposition 2.2, the minimum number of edges separating s from t in G is equal to the minimum number of vertices separating S from T in $L(G)$. By Proposition 2.3, the maximum number of pairwise edge-disjoint s-t paths in G is equal to the maximum number of pairwise disjoint S-T paths in G. By Menger's theorem (vertex version), the minimum number of vertices separating S from T in $L(G)$ is equal to the maximum number of pairwise disjoint S-T paths in G. We now deduce that the minimum number of edges separating s from t in G is equal to the maximum number of pairwise edge-disjoint s-t paths in G. This completes the argument. $\square$

The global version of Menger's theorem. *Let G be a graph on at least two vertices, and let $k, \ell \geq 0$ be integers.*

(a) G is k-connected if and only if for all distinct $s, t \in V(G)$, there are k pairwise internally disjoint s-t paths in G.

(b) G is ℓ-edge-connected if and only if for all distinct $s, t \in E(G)$, there are ℓ pairwise edge-disjoint s-t paths in G.

Proof. HW. □