NDMI011: Combinatorics and Graph Theory 1

Lecture #6

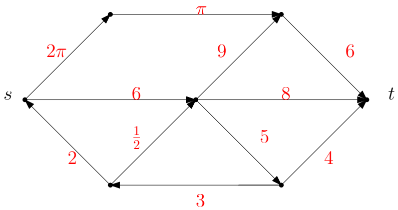
Flows and cuts in networks

Irena Penev

November 2, 2020

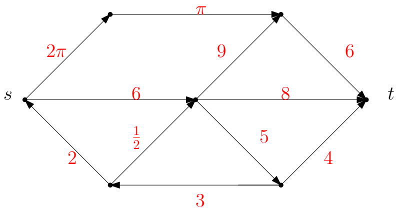
Definition

A *network* is an ordered four-tuple (G, s, t, c), where G is an oriented graph, s and t are two distinct vertices of this graph (called the *source* and *sink*, respectively), and $c : E(G) \to [0, +\infty)$ is a function, called the *capacity function*. The *capacity* of an edge $e \in E(G)$ is the number $c(e)$.



Definition

A *network* is an ordered four-tuple (G, s, t, c), where G is an oriented graph, s and t are two distinct vertices of this graph (called the *source* and *sink*, respectively), and $c : E(G) \to [0, +\infty)$ is a function, called the *capacity function*. The *capacity* of an edge $e \in E(G)$ is the number $c(e)$.

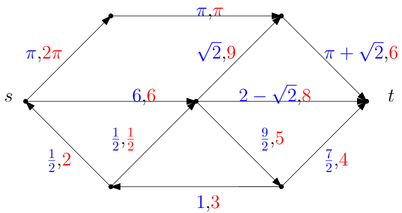


- Networks can be used to model, for example, a system of pipes used to transport some resource, such as water or oil; capacities would be the number of units of volume that a given pipe can transport per unit time.

Definition

A *feasible flow* (or simply *flow*) in a network (G, s, t, c) is a function $f : E(G) \to [0, +\infty)$ s.t.:

- $f(e) \leq c(e)$ for all $e \in E(G)$;
- for all $v \in V(G) \setminus \{s, t\}$, we have $\sum_{(x,v) \in E(G)} f(x, v) = \sum_{(v,y) \in E(G)} f(v, y)$.

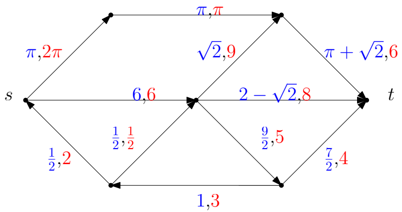


Definition

The *value* of a flow f in a network (G, s, t, c) is

$$val(f) = \Big(\sum_{(s,x)\in E(G)} f(s,x)\Big) - \Big(\sum_{(x,s)\in E(G)} f(x,s)\Big).$$

A *maximum flow* in (G, s, t, c) is a flow f^* that has maximum value, i.e. one that satisfies $val(f) \leq val(f^*)$ for all flows f.



- The value of the flow above is $\pi + 6 - \frac{1}{2} = \frac{11}{2} + \pi$.

Theorem 1.1

Every network (G, s, t, c) has a maximum flow.

Proof. Omitted.

Theorem 1.1

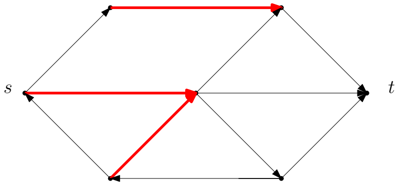
Every network (G, s, t, c) has a maximum flow.

Proof. Omitted.

- Theorem 1.1 should seem plausible, but the proof is not obvious (since the number of flows is, typically, infinite).
- The proof relies on certain results from analysis, which we omit.

Definition

An s, t-*cut*, or simply *cut*, in a network (G, s, t, c) is a set $R \subseteq E(G)$ such that $G \setminus R$ contains no directed path from s to t. The *capacity* of the cut R is $c(R) = \sum_{e \in R} c(e)$.



Max-flow min-cut theorem

The maximum value of a flow in a network is equal to the minimum capacity of a cut in that network.

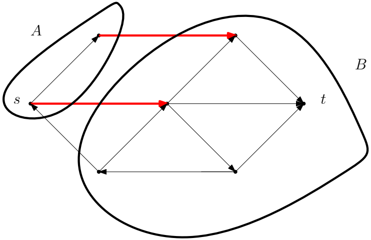
- For a network (G, s, t, c), a flow f in that network, and a set of edges $R \subseteq E(G)$, we write

$$c(R) = \sum_{e \in R} c(e) \qquad \text{and} \qquad f(R) = \sum_{e \in R} f(e).$$

- For a network (G, s, t, c), a flow f in that network, and a set of edges $R \subseteq E(G)$, we write

$$c(R) = \sum_{e \in R} c(e) \qquad \text{and} \qquad f(R) = \sum_{e \in R} f(e).$$

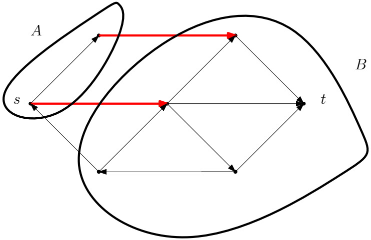
- For a directed graph G and disjoint sets $A, B \subseteq V(G)$, we set $S(A, B) = \{(a, b) \in E(G) \mid a \in A, b \in B\}$.



- For a network (G, s, t, c), a flow f in that network, and a set of edges $R \subseteq E(G)$, we write

$$c(R) = \sum_{e \in R} c(e) \qquad \text{and} \qquad f(R) = \sum_{e \in R} f(e).$$

- For a directed graph G and disjoint sets $A, B \subseteq V(G)$, we set $S(A, B) = \{(a, b) \in E(G) \mid a \in A, b \in B\}$.



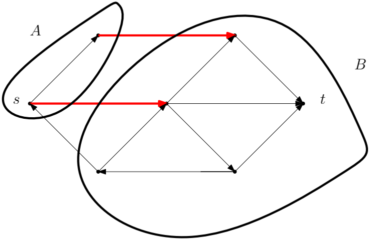
- For a network (G, s, t, c), disjoint sets $A, B \subseteq V(G)$, and a flow f, we write

$$c(A, B) = c(S(A, B)) \qquad \text{and} \qquad f(A, B) = f(S(A, B)).$$

Proposition 2.1

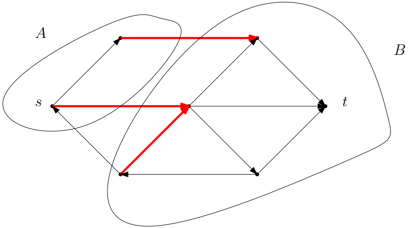
Let (G, s, t, c) be a network, and let (A, B) be a partition of $V(G)$ such that $s \in A$ and $t \in B$. Then $S(A, B)$ is a cut in (G, s, t, c).

Proof. Lecture Notes.



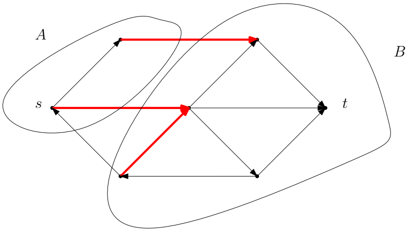
Proposition 2.2

Let (G, s, t, c) be a network, and let R be a cut in this network. Then there exists a partition (A, B) of $V(G)$ such that $s \in A$, $t \in B$, and $S(A, B) \subseteq R$.



Proposition 2.2

Let (G, s, t, c) be a network, and let R be a cut in this network. Then there exists a partition (A, B) of $V(G)$ such that $s \in A$, $t \in B$, and $S(A, B) \subseteq R$.



Proof (outline). Let A be the set of all vertices $v \in V(G)$ such that $G \setminus R$ contains a directed path from s to v, and set $B = V(G) \setminus A$.

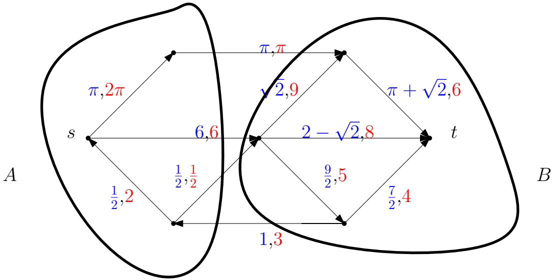
Lemma 2.3

Let f be a flow in a network (G, s, t, c), and let (A, B) be a partition of $V(G)$ such that $s \in A$ and $t \in B$. Then $val(f) = f(A, B) - f(B, A)$. In particular,[a] we have that

$$val(f) = \Big(\sum_{(x,t) \in E(G)} f(x, t) \Big) - \Big(\sum_{(t,x) \in E(G)} f(t, x) \Big).$$

[a] This happens if we take $A = V(G) \setminus \{t\}$ and $B = \{t\}$.

Proof. Lecture Notes.



$val(f) = \frac{11}{2} + \pi$

Corollary 2.4

Let f be a flow in a network (G, s, t, c), and let R be a cut. Then $val(f) \leq c(R)$.

Proof.

Corollary 2.4

Let f be a flow in a network (G, s, t, c), and let R be a cut. Then $val(f) \leq c(R)$.

Proof. By Proposition 2.2, there exists a partition (A, B) of $V(G)$ such that $s \in A$, $t \in B$, and $S(A, B) \subseteq R$.

Corollary 2.4

Let f be a flow in a network (G, s, t, c), and let R be a cut. Then $val(f) \leq c(R)$.

Proof. By Proposition 2.2, there exists a partition (A, B) of $V(G)$ such that $s \in A$, $t \in B$, and $S(A, B) \subseteq R$.

Corollary 2.4

Let f be a flow in a network (G, s, t, c), and let R be a cut. Then $val(f) \leq c(R)$.

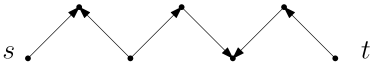
Proof. By Proposition 2.2, there exists a partition (A, B) of $V(G)$ such that $s \in A$, $t \in B$, and $S(A, B) \subseteq R$. Then

$$\begin{aligned} val(f) &= f(A,B) - f(B,A) && \text{by Lemma 2.3} \\ &\leq f(A,B) && \text{because } f(e) \geq 0 \ \forall e \in E(G) \\ &\leq c(A,B) && \text{because } f(e) \leq c(e) \ \forall e \in E(G) \\ &\leq c(R) && \text{because } S(A,B) \subseteq R \text{ and} \\ & && \text{and } c(e) \geq 0 \ \forall e \in E(G) \end{aligned}$$

which is what we needed to show.

Definition

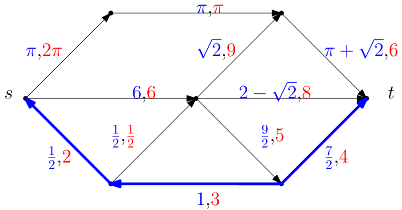
An (s, t)-*path* in a network (G, s, t, c) is a sequence $v_0, v_1, \ldots, v_\ell$ of vertices of G such that $v_0 = s$, $v_\ell = t$, and for all $i \in \{0, \ldots, \ell - 1\}$, we have that one of (v_i, v_{i+1}) and (v_{i+1}, v_i) belongs to $E(G)$.



Definition

Given a flow f in the network (G, s, t, c), an (s, t)-path $v_0, v_1, \dots, v_\ell$ in (G, s, t, c) is said to be an *f-augmenting path* if the following two conditions are satisfied:

- for all $i \in \{1, \dots, \ell - 1\}$ such that $(v_i, v_{i+1}) \in E(G)$, we have that $f(v_i, v_{i+1}) < c(v_i, v_{i+1})$;
- for all $i \in \{1, \dots, \ell - 1\}$ such that $(v_{i+1}, v_i) \in E(G)$, we have that $f(v_{i+1}, v_i) > 0$.



Lemma 2.5

Let f be a flow in a network (G, s, t, c). Then f is a maximum flow if and only if there does not exist an f-augmenting path in (G, s, t, c). Furthermore, if f is a maximum flow, then there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

Proof.

Lemma 2.5

Let f be a flow in a network (G, s, t, c). Then f is a maximum flow if and only if there does not exist an f-augmenting path in (G, s, t, c). Furthermore, if f is a maximum flow, then there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

Proof. It suffices the prove the following two statements:

(a) If there exists an f-augmenting path in (G, s, t, c), then f is not a maximum flow in (G, s, t, c).

(b) If there does not exist an f-augmenting path in (G, s, t, c), then f is a maximum flow in (G, s, t, c), and furthermore, there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

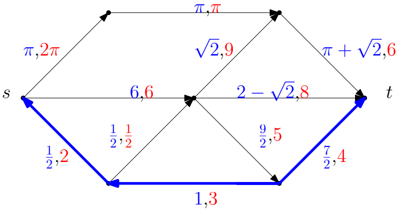
(a) If there exists an f-augmenting path in (G, s, t, c), then f is not a maximum flow in (G, s, t, c).

Proof of (a). Suppose that $v_0, \dots, v_\ell$ (with $v_0 = s$ and $v_\ell = t$) an f-augmenting path in (G, s, t, c).

(a) If there exists an f-augmenting path in (G, s, t, c), then f is not a maximum flow in (G, s, t, c).

Proof of (a). Suppose that $v_0, \dots, v_\ell$ (with $v_0 = s$ and $v_\ell = t$) an f-augmenting path in (G, s, t, c). Now, set

- $\varepsilon_1 = \min\Big(\{c(v_i, v_{i+1}) - f(v_i, v_{i+1}) \mid 0 \leq i \leq \ell - 1,$ $(v_i, v_{i+1}) \in E(G)\} \cup \{\infty\}\Big)$;
- $\varepsilon_2 = \min\Big(\{f(v_{i+1}, v_i) \mid 0 \leq i \leq \ell - 1, (v_{i+1}, v_i) \in E(G)\}$ $\cup\{\infty\}\Big)$;
- $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.



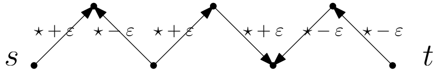
$\varepsilon_1 = \frac{1}{2}$

$\varepsilon_2 = \frac{1}{2}$

$\varepsilon = \frac{1}{2}$

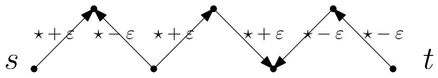
We now define a new flow f' as follows:

- $f'(v_i, v_{i+1}) = f(v_i, v_{i+1}) + \varepsilon$ for all $i \in \{0, \ldots, \ell - 1\}$ such that $(v_i, v_{i+1}) \in E(G)$;
- $f'(v_{i+1}, v_i) = f(v_{i+1}, v_i) - \varepsilon$ for all $i \in \{0, \ldots, \ell - 1\}$ such that $(v_{i+1}, v_i) \in E(G)$;
- $f'(e) = f(e)$ for all other edges e.



We now define a new flow f' as follows:

- $f'(v_i, v_{i+1}) = f(v_i, v_{i+1}) + \varepsilon$ for all $i \in \{0, \ldots, \ell - 1\}$ such that $(v_i, v_{i+1}) \in E(G)$;
- $f'(v_{i+1}, v_i) = f(v_{i+1}, v_i) - \varepsilon$ for all $i \in \{0, \ldots, \ell - 1\}$ such that $(v_{i+1}, v_i) \in E(G)$;
- $f'(e) = f(e)$ for all other edges e.



Then $val(f) < val(f')$, and so f is not a maximum flow. This proves (a).

(b) If there does not exist an f-augmenting path in (G, s, t, c), then f is a maximum flow in (G, s, t, c), and furthermore, there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

Proof of (b). Suppose that (G, s, t, c) does not admit an f-augmenting path.

(b) If there does not exist an f-augmenting path in (G, s, t, c), then f is a maximum flow in (G, s, t, c), and furthermore, there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

Proof of (b). Suppose that (G, s, t, c) does not admit an f-augmenting path. Let A to be the set of all vertices $v \in V(G)$ such that there exists an f-augmenting path from s to v. Let $B = V(G) \setminus A$.

(b) If there does not exist an f-augmenting path in (G, s, t, c), then f is a maximum flow in (G, s, t, c), and furthermore, there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

Proof of (b). Suppose that (G, s, t, c) does not admit an f-augmenting path. Let A to be the set of all vertices $v \in V(G)$ such that there exists an f-augmenting path from s to v. Let $B = V(G) \setminus A$. Clearly, $s \in A$ and $t \notin A$.

(b) If there does not exist an f-augmenting path in (G, s, t, c), then f is a maximum flow in (G, s, t, c), and furthermore, there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

Proof of (b). Suppose that (G, s, t, c) does not admit an f-augmenting path. Let A to be the set of all vertices $v \in V(G)$ such that there exists an f-augmenting path from s to v. Let $B = V(G) \setminus A$. Clearly, $s \in A$ and $t \notin A$. Then $f(A, B) = c(A, B)$ and $f(B, A) = 0$, and so

$$\begin{aligned} val(f) &= f(A, B) - f(B, A) && \text{by Lemma 2.3} \\ &= c(A, B) && \text{because } f(A, B) = c(A, B) \\ & && \text{and } f(B, A) = 0 \end{aligned}$$

(b) If there does not exist an f-augmenting path in (G, s, t, c), then f is a maximum flow in (G, s, t, c), and furthermore, there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

Proof of (b). Suppose that (G, s, t, c) does not admit an f-augmenting path. Let A to be the set of all vertices $v \in V(G)$ such that there exists an f-augmenting path from s to v. Let $B = V(G) \setminus A$. Clearly, $s \in A$ and $t \notin A$. Then $f(A, B) = c(A, B)$ and $f(B, A) = 0$, and so

$$\begin{aligned} val(f) &= f(A, B) - f(B, A) && \text{by Lemma 2.3} \\ &= c(A, B) && \text{because } f(A, B) = c(A, B) \text{ and } f(B, A) = 0 \end{aligned}$$

It now follows from Corollary 2.4 that f is a maximum flow in (G, s, t, c).

(b) If there does not exist an f-augmenting path in (G, s, t, c), then f is a maximum flow in (G, s, t, c), and furthermore, there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$.

Proof of (b). Suppose that (G, s, t, c) does not admit an f-augmenting path. Let A to be the set of all vertices $v \in V(G)$ such that there exists an f-augmenting path from s to v. Let $B = V(G) \setminus A$. Clearly, $s \in A$ and $t \notin A$. Then $f(A, B) = c(A, B)$ and $f(B, A) = 0$, and so

$$\begin{aligned} val(f) &= f(A,B) - f(B,A) && \text{by Lemma 2.3} \\ &= c(A,B) && \text{because } f(A,B) = c(A,B) \\ & && \text{and } f(B,A) = 0 \end{aligned}$$

It now follows from Corollary 2.4 that f is a maximum flow in (G, s, t, c). Furthermore, by Proposition 2.1, we know that $R := S(A, B)$ is a cut, and by what we just showed, $val(f) = c(A, B) = c(R)$.

Max-flow min-cut theorem

The maximum value of a flow in a network is equal to the minimum capacity of a cut in that network.

Proof. Let (G, s, t, c) be a network, and let f be a maximum flow in it (the existence of such a flow is guaranteed by Theorem 1.1). By Lemma 2.5, there exists a cut R in (G, s, t, c) such that $val(f) = c(R)$. Furthermore, for any cut R' in (G, s, t, c), Corollary 2.4 guarantees that $val(f) \leq c(R')$, and consequently, $c(R) \leq c(R')$; thus, R is a cut of minimum capacity in (G, s, t, c).

- Our next goal is to show how to find a maximum flow and a minimum cut in a network.

- Our next goal is to show how to find a maximum flow and a minimum cut in a network.
- The idea is to repeatedly find augmenting paths and update the flow (increasing its value).

- Our next goal is to show how to find a maximum flow and a minimum cut in a network.
- The idea is to repeatedly find augmenting paths and update the flow (increasing its value).
- When no augmenting path exists, we instead find a cut whose capacity is equal to the value of our flow, which (by Corollary 2.4) guarantees that this cut is of minimum capacity.

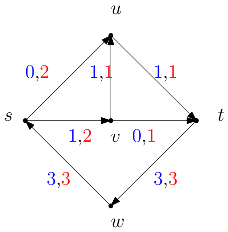
Suppose that f is a flow in a network (G, s, t, c). We now either find an f-augmenting path in (G, s, t, c), or we find a cut whose capacity is $val(f)$, as follows:

1. Set $A := \{s\}$.
2. While $t \notin A$:
    1. Either find vertices $x \in A$ and $y \in V(G) \setminus A$ such that
        - $(x, y) \in E(G)$ and $f(x, y) < c(x, y)$, or
        - $(y, x) \in E(G)$ and $f(y, x) > 0$,

       or determine that such x and y do not exist.
    2. If we found x and y, then we set backpoint$(y) = x$, and we update $A := A \cup \{y\}$.
    3. Otherwise, we stop and return the cut $S(A, V(G) \setminus A)$.[1]
3. Construct an f-augmenting path by following backpoints starting from t, and return this path.

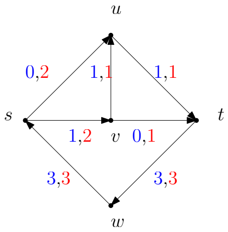
---

[1] In this case, an argument analogous to the proof of Lemma 2.5 guarantees that $c(A, V(G) \setminus A) = val(f)$.

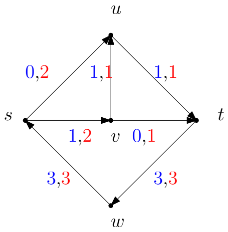


Example 3.1

Consider the flow f in the network (G, s, t, c) as in the figure above. Either find an f-augmenting path, or find a cut whose capacity is $val(f)$.

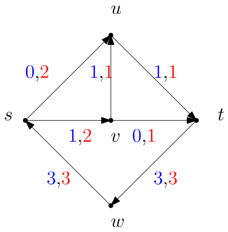


Solution. We begin with $A = \{s\}$. We now iterate several times.



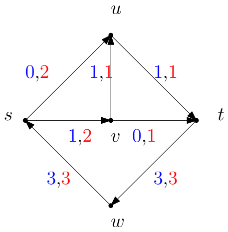
Solution. We begin with $A = \{s\}$. We now iterate several times.

1. We select $s \in A$ and $u \in V(G) \setminus A$, and we set $A := \{s, u\}$ and backpoint$(u) = s$.



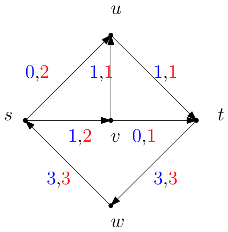
Solution. We begin with $A = \{s\}$. We now iterate several times.

1. We select $s \in A$ and $u \in V(G) \setminus A$, and we set $A := \{s, u\}$ and backpoint$(u) = s$.
2. We select $s \in A$ and $w \in V(G) \setminus A$, and we set $A := \{s, u, w\}$ and backpoint$(w) = s$.



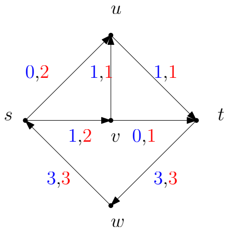
Solution. We begin with $A = \{s\}$. We now iterate several times.

1. We select $s \in A$ and $u \in V(G) \setminus A$, and we set $A := \{s, u\}$ and backpoint$(u) = s$.
2. We select $s \in A$ and $w \in V(G) \setminus A$, and we set $A := \{s, u, w\}$ and backpoint$(w) = s$.
3. We select $u \in A$ and $v \in V(G) \setminus A$, and we set $A := \{s, u, w, v\}$ and backpoint$(v) = u$.



Solution. We begin with $A = \{s\}$. We now iterate several times.

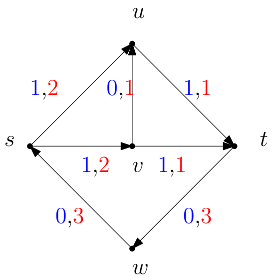
1. We select $s \in A$ and $u \in V(G) \setminus A$, and we set $A := \{s, u\}$ and backpoint$(u) = s$.
2. We select $s \in A$ and $w \in V(G) \setminus A$, and we set $A := \{s, u, w\}$ and backpoint$(w) = s$.
3. We select $u \in A$ and $v \in V(G) \setminus A$, and we set $A := \{s, u, w, v\}$ and backpoint$(v) = u$.
4. We select $v \in A$ and $t \in V(G) \setminus A$, and we set $A := \{s, u, w, v, t\}$ and backpoint$(t) = v$.



Solution. We begin with $A = \{s\}$. We now iterate several times.

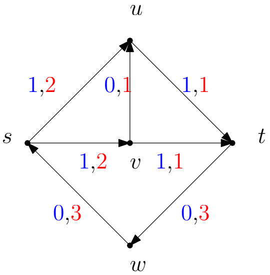
1. We select $s \in A$ and $u \in V(G) \setminus A$, and we set $A := \{s, u\}$ and backpoint$(u) = s$.
2. We select $s \in A$ and $w \in V(G) \setminus A$, and we set $A := \{s, u, w\}$ and backpoint$(w) = s$.
3. We select $u \in A$ and $v \in V(G) \setminus A$, and we set $A := \{s, u, w, v\}$ and backpoint$(v) = u$.
4. We select $v \in A$ and $t \in V(G) \setminus A$, and we set $A := \{s, u, w, v, t\}$ and backpoint$(t) = v$.

We now reconstruct our f-augmenting path: s, u, v, t. (It is easy to see that this really is an f-augmenting path.)

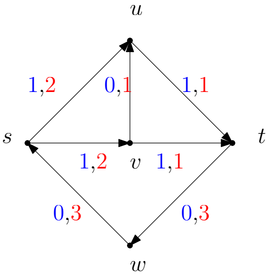


Example 3.2

Consider the flow f in the network (G, s, t, c) as in the figure above. Either find an f-augmenting path, or find a cut whose capacity is $val(f)$.

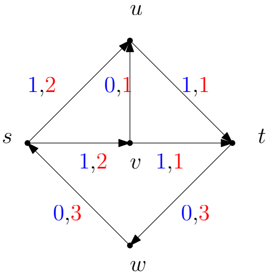


Solution. We begin with $A = \{s\}$. We now iterate several times.



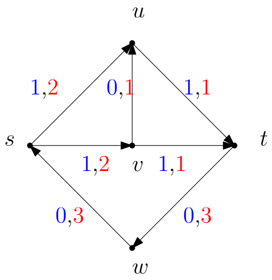
Solution. We begin with $A = \{s\}$. We now iterate several times.

1. We select $s \in A$ and $u \in V(G) \setminus A$, and we set $A := \{s, u\}$ and $\text{backpoint}(u) = s$.



Solution. We begin with $A = \{s\}$. We now iterate several times.

1. We select $s \in A$ and $u \in V(G) \setminus A$, and we set $A := \{s, u\}$ and $\text{backpoint}(u) = s$.
2. We select $s \in A$ and $v \in V(G) \setminus A$, and we set $A := \{s, u, v\}$ and $\text{backpoint}(v) = s$.



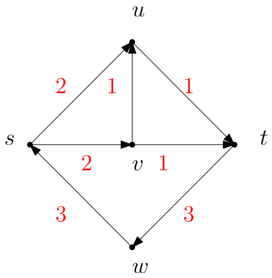
Solution. We begin with $A = \{s\}$. We now iterate several times.

1. We select $s \in A$ and $u \in V(G) \setminus A$, and we set $A := \{s, u\}$ and backpoint$(u) = s$.
2. We select $s \in A$ and $v \in V(G) \setminus A$, and we set $A := \{s, u, v\}$ and backpoint$(v) = s$.

There are now no further vertices that we can select, and $t \notin A$. We now see that $S(A, V(G) \setminus A) = \{(u, t), (v, t)\}$ is a cut whose capacity is 2, which is precisely equal to $val(f)$.

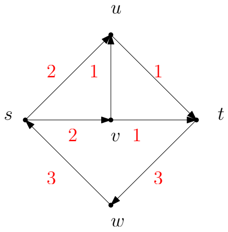
We now describe the Ford-Fulkerson algorithm, which finds a maximum flow in a network (G, s, t, c). Its steps are as follows:

1. Set $f(e) := 0$ for all $e \in E(G)$.
2. While there exists an f-augmenting path in the network:
    1. Find an f-augmenting path $v_0, \dots, v_\ell$ (with $v_0 = s$ and $v_\ell = t$).
    2. Set
        - $\varepsilon_1 = \min\Big(\{c(v_i, v_{i+1}) - f(v_i, v_{i+1}) \mid 0 \leq i \leq \ell - 1, (v_i, v_{i+1}) \in E(G)\} \cup \{\infty\}\Big)$;
        - $\varepsilon_2 = \min\Big(\{f(v_{i+1}, v_i) \mid 0 \leq i \leq \ell - 1, (v_{i+1}, v_i) \in E(G)\} \cup \{\infty\}\Big)$;
        - $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.
    3. Update f as follows:
        - $f(v_i, v_{i+1}) := f(v_i, v_{i+1}) + \varepsilon$ for all $i \in \{0, \dots, \ell - 1\}$ such that $(v_i, v_{i+1}) \in E(G)$;
        - $f(v_{i+1}, v_i) := f(v_{i+1}, v_i) - \varepsilon$ for all $i \in \{0, \dots, \ell - 1\}$ such that $(v_{i+1}, v_i) \in E(G)$.
3. Return f.

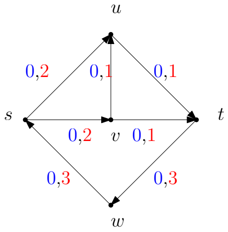


Example 3.3

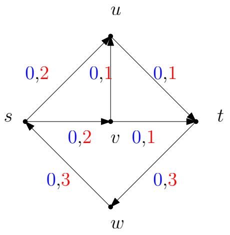
Find a maximum flow and an a cut of minimum capacity in the network represented in the figure above.



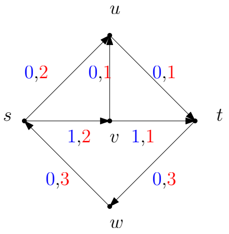
Solution. We first set $f(e) = 0$ for all $e \in E(G)$.

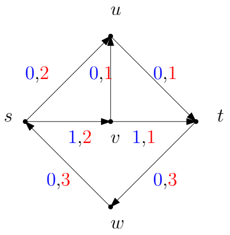


We now iterate several times.

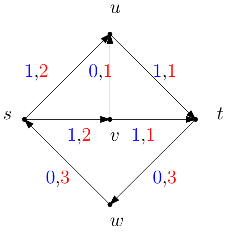


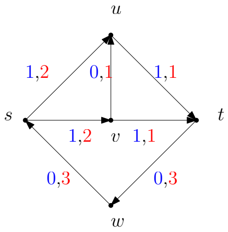
(1) We find an augmenting path s, v, t, we get $\varepsilon = 1$, and we update f as in the picture below.



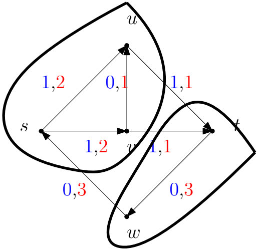


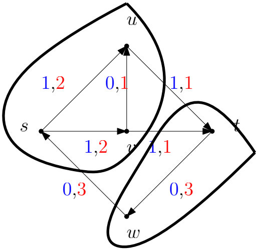
(2) We find an augmenting path s, u, t, we get $\varepsilon = 1$, and we update f as in the picture below.





(3) We find a cut $S(\{s, u, v\}, \{w, t\}) = \{(u, t), (v, t)\}$ of capacity is 2, which is precisely equal to $val(f)$.





- The flow f (blue) is a maximum flow.
- The cut $S(\{s, u, v\}, \{w, t\}) = \{(u, t), (v, t)\}$ is a minimum capacity cut.