NDMI011: Combinatorics and Graph Theory 1

Lecture #5

Finite projective planes (part II)

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3. An algebraic construction of a (not necessarily finite) projective plane.

Part I: A brief review of the previous lecture

Definition

A *projective plane* is a set system $(X, \mathcal{P})$[a] that satisfies the following three properties:

(P0) there exists a 4-element subset $Q \subseteq X$ s.t. every $P \in \mathcal{P}$ satisfies $|P \cap Q| \leq 2$;

(P1) all distinct $P_1, P_2 \in \mathcal{P}$ satisfy $|P_1 \cap P_2| = 1$;

(P2) for all distinct $x_1, x_2 \in X$, there exists a unique $P \in \mathcal{P}$ s.t. $x_1, x_2 \in P$.

Elements of X are called *points*, and elements of $\mathcal{P}$ are called *lines* of the projective plane $(X, \mathcal{P})$.

A projective plane $(X, \mathcal{P})$ is *finite* if X is finite.

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[a] This means that X is a set and $\mathcal{P} \subseteq \mathscr{P}(X)$, where $\mathscr{P}(X)$ is the power set (i.e. the set of all subsets) of X.

Proposition 1.2 from Lecture Notes 4

Let $(X, \mathcal{P})$ be a finite projective plane. Then all lines in $\mathcal{P}$ have the same number of points.

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The *order* of a finite projective plane $(X, \mathcal{P})$ is the number $|P| - 1$, where P is any line in $\mathcal{P}$.[a]

[a] So, if $(X, \mathcal{P})$ is a finite projective plane of order n, then each line in $\mathcal{P}$ contains exactly $n + 1$ points.

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Proposition 1.3 from Lecture Notes 4

The order of any finite projective plane is at least two.

Theorem 1.4 from Lecture Notes 4

Let $(X, \mathcal{P})$ be a finite projective plane of order n. Then all the following hold:

(a) for each point $x \in X$, exactly $n+1$ lines in $\mathcal{P}$ pass through x;

(b) $|X| = n^2 + n + 1$;

(c) $|\mathcal{P}| = n^2 + n + 1$.

Part II: A construction of a finite projective plane from orthogonal Latin squares

Definition

For a positive integer n, an $n \times n$ *Latin square* is an $n \times n$ array (or matrix) whose entries are numbers $1, \ldots, n$, and in which each number $1, \ldots, n$ occurs exactly once in each row and in each column.

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

Figure: Two 3×3 Latin squares.

Definition

Two $n \times n$ Latin squares, say $[a_{i,j}]_{n\times n}$ and $[b_{i,j}]_{n\times n}$, are *orthogonal* if each entry of the matrix matrix obtained by superimposing A on B, i.e. of the matrix $[(a_{i,j}, b_{i,j})]_{n\times n}$, is different.

- The red and the blue Latin square (below) are orthogonal.

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

$(1,1)$	$(2,2)$	$(3,3)$
$(2,3)$	$(3,1)$	$(1,2)$
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- An $n \times n$ matrix has n^2 entries.
- The Cartesian product $\{1, \ldots, n\} \times \{1, \ldots, n\}$ has exactly n^2 elements.
- So, two $n \times n$ Latin squares are orthogonal if and only if each element of $\{1, \ldots, n\} \times \{1, \ldots, n\}$ appears exactly once in the matrix obtained by superimposing the two $n \times n$ Latin squares.

- For a positive integer n, a Latin square $A = [a_{i,j}]_{n\times n}$ and a permutation π of the set $\{1, \dots, n\}$, we set $\pi(A) = [\pi(a_{i,j})]_{n\times n}$; obviously, $\pi(A)$ is a Latin square.
- For example, if

$A =$

1	3	2
3	2	1
2	1	3

and if $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, then

$\pi(A) =$

1	2	3
2	3	1
3	1	2

.

Proposition 2.1

Let $A = [a_{i,j}]_{n \times n}$ and $B = [b_{i,j}]_{n \times n}$ be orthogonal $n \times n$ Latin squares, and let π_A, π_B be permutations of the set $\{1, \ldots, n\}$. Then $\pi_A(A)$ and $\pi_B(B)$ are orthogonal Latin squares.

Proof. Obvious.

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Let $n \geq 2$ be an integer, and let M be a set of pairwise orthogonal $n \times n$ Latin squares. Then $|M| \leq n - 1$.

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For each $i \in \{1, \ldots, t\}$, we let π_i be the permutation of $\{1, \ldots, n\}$ that transforms the first row of A_i into $1, \ldots, n$, and let $A_i' = \pi_i(A_i)$.

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For each $i \in \{1, \ldots, t\}$, we let π_i be the permutation of $\{1, \ldots, n\}$ that transforms the first row of A_i into $1, \ldots, n$, and let $A_i' = \pi_i(A_i)$. By Proposition 2.1, $A_1', \ldots, A_t'$ are pairwise orthogonal.

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Let $n \geq 2$ be an integer, and let M be a set of pairwise orthogonal $n \times n$ Latin squares. Then $|M| \leq n-1$.

Proof (outline, continued). For distinct $i, j \in \{1, \dots, t\}$, the matrix obtained by superimposing A'_i onto A'_j looks like this:

$(1,1)$	$(2,2)$	$\dots$	(n,n)
(?,?)	(?,?)	$\dots$	(?,?)
$\vdots$	$\vdots$	$\ddots$	$\vdots$
(?,?)	(?,?)	$\dots$	(?,?)

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So, no A'_i can have 1 in the $(2,1)$-th spot, and no two of $A'_1, \ldots, A'_t$ can have the same $(2,1)$-th entry.

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Theorem 2.3

Let $n \geq 2$ be an integer. Then the following are equivalent:

(a) $\exists$ a finite projective plane of order n;

(b) $\exists$ a collection of $n-1$ pairwise orthogonal $n \times n$ Latin squares.

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Proof of "(b) $\Longrightarrow$ (a)" (outline).

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Proof of "(b) $\Longrightarrow$ (a)" (outline). Assume that (b) is true, and let $L_1, \ldots, L_{n-1}$ be pairwise orthogonal $n \times n$ Latin squares. We will give a construction of the corresponding finite projective plane of order n (proof that it works: HW).

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- There are $n^2 + n + 1$ points:
  - points r and s;
  - points ℓ_i for $i \in \{1, \dots, n-1\}$;
  - points $x_{i,j}$ for $i, j \in \{1, \dots, n\}$.

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  - points r and s;
  - points ℓ_i for $i \in \{1, \ldots, n-1\}$;
  - points $x_{i,j}$ for $i, j \in \{1, \ldots, n\}$.
- There are $n^2 + n + 1$ lines:
  - line B
  - lines R_i for $i \in \{1, \ldots, n\}$;
  - lines S_j for $j \in \{1, \ldots, n\}$;
  - lines L_i^j for $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n\}$.

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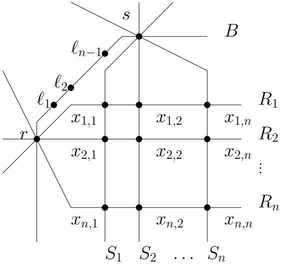
Proof of "(b) $\Longrightarrow$ (a)" (outline, continued). Reminder: $L_1, \ldots, L_{n-1}$ are pairwise orthogonal $n \times n$ Latin squares.



- $L_i^j = \{\ell_i\} \cup \{x_{p,q} \mid \text{the } (p,q)\text{-th entry of } L_i \text{ is } j\}$, for $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n\}$.

$$L_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} \qquad L_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 1 & 2 \\ \hline 2 & 3 & 1 \\ \hline \end{array}$$

- For example, for L_1, L_2 as above, we get points

$$r, s, \ell_1, \ell_2, x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3}.$$

and lines

- $B = \{r, s, \ell_1, \ell_2\}$;
- $R_1 = \{r, x_{1,1}, x_{1,2}, x_{1,3}\}$;
- $R_2 = \{r, x_{2,1}, x_{2,2}, x_{2,3}\}$;
- $R_3 = \{r, x_{3,1}, x_{3,2}, x_{3,3}\}$;
- $S_1 = \{s, x_{1,1}, x_{2,1}, x_{3,1}\}$;
- $S_2 = \{s, x_{1,2}, x_{2,2}, x_{3,2}\}$;
- $S_3 = \{s, x_{1,3}, x_{2,3}, x_{3,3}\}$;
- $L_1^1 = \{\ell_1, x_{1,1}, x_{2,3}, x_{3,2}\}$;
- $L_1^2 = \{\ell_1, x_{1,2}, x_{2,1}, x_{3,3}\}$;
- $L_1^3 = \{\ell_1, x_{1,3}, x_{2,2}, x_{3,1}\}$;
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Part III: An algebraic construction of a (not necessarily finite) projective plane

- Let $\mathbb{F}$ be any field.

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  - 0 and 1 are, respectively, the additive and multiplicative identity in $\mathbb{F}$.

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- Let $T := \mathbb{F}^3 \setminus \{(0,0,0)\}$.
- For $(x_1, y_1, z_1), (x_2, y_2, z_2) \in T$: $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if there exists a scalar $\lambda \in \mathbb{F} \setminus \{0\}$ s.t. $(x_2, y_2, z_2) = \lambda(x_1, y_1, z_1)$, i.e. $x_2 = \lambda x_1$, $y_2 = \lambda y_1$, $z_2 = \lambda z_1$.

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  - Obviously, $\sim$ is an equivalence relation on T.
  - The equivalence class of $(x, y, z) \in T$ is $\overline{(x, y, z)} = \{(\lambda x, \lambda y, \lambda z) \mid \lambda \in \mathbb{F} \setminus \{0\}\}$.

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- For each $(a, b, c) \in T$: $P(a, b, c) = \{\overline{(x, y, z)} \mid (x, y, z) \in T, ax + by + cz = 0\}$.

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  - For all $(a_1, b_1, c_1), (a_2, b_2, c_2) \in T$, we have that $P(a_1, b_1, c_1) = P(a_2, b_2, c_2)$ if and only if $(a_1, b_1, c_1) \sim (a_2, b_2, c_2)$.

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- Lines are the sets $P(a, b, c)$ with $(a, b, c) \in T$.

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Proof. Reminder: for $(a, b, c) \in T$,
$P(a, b, c) = \{\overline{(x, y, z)} \mid (x, y, z) \in T, ax + by + cz = 0\}$.

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For each field $\mathbb{F}$, $\mathbb{F}P^2$ is a projective plane.

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First, we check that (P0) is satisfied for

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We note that each of the following four matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has rank three.

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The proof of (P2) is similar.

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Theorem 1.4 from Lecture Notes 4

Let $(X, \mathcal{P})$ be a finite projective plane of order n. Then all the following hold:

(a) for each point $x \in X$, exactly $n+1$ lines in $\mathcal{P}$ pass through x;

(b) $|X| = n^2 + n + 1$;

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If $\mathbb{F}$ is a finite field, with $|\mathbb{F}| = n$, then $\mathbb{F}P^2$ is a finite projective plane of order n.

Proof. By Theorem 3.1, $\mathbb{F}P^2$ is a projective plane.

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Proof. By Theorem 3.1, $\mathbb{F}P^2$ is a projective plane. Since $\mathbb{F}$ is finite, so is $\mathbb{F}P^2$.

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Proof. By Theorem 3.1, $\mathbb{F}P^2$ is a projective plane. Since $\mathbb{F}$ is finite, so is $\mathbb{F}P^2$. In view of Theorem 1.4 from Lecture Notes 4, it suffices to show that $\mathbb{F}P^2$ has precisely $n^2 + n + 1$ points.

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If $\mathbb{F}$ is a finite field, with $|\mathbb{F}| = n$, then $\mathbb{F}P^2$ is a finite projective plane of order n.

Proof (continued). Reminder: WTS $\mathbb{F}P^2$ has $n^2 + n + 1$ points.

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- if $z \neq 0$, then $(x, y, z) \sim (z^{-1}x, z^{-1}y, 1)$;
- if $z = 0$ and $y \neq 0$, then $(x, y, z) \sim (y^{-1}x, 1, 0)$;
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There are n^2 triples of the form $(x, y, 1)$ in T; there are n triples of the form $(x, 1, 0)$ in T; and there is one triple $(1, 0, 0)$ in T.

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If $\mathbb{F}$ is a finite field, with $|\mathbb{F}| = n$, then $\mathbb{F}P^2$ is a finite projective plane of order n.

- It is well-known that for all integers $n \geq 2$, there exists a field of size n if and only if n is a power of a prime (that is, if and only if there exist a prime number p and a positive integer k s.t. $n = p^k$).

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- It is, however, known that there are no finite projective planes of order 6 or 10.

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- It is, however, known that there are no finite projective planes of order 6 or 10.
- It is **not** known whether there are finite projective planes of order 12.