NDMI011: Combinatorics and Graph Theory 1

Lecture #4

Finite projective planes (part I)

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A *finite projective plane* is set system $(X, \mathcal{P})$ s.t. X is a finite, and the following three properties are satisfied:

(P0) there exists a 4-element subset $Q \subseteq X$ s.t. every $P \in \mathcal{P}$ satisfies $|P \cap Q| \leq 2$;

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- For distinct points $a, b \in X$, we denote by $\overline{ab}$ the unique line in $\mathcal{P}$ that contains a and b (the existence and uniqueness of such a line follow from (P2)).

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- (P2) is the same as for points and lines in the Euclidean plane.
- But (P1) is different! There are no "parallel lines" in a finite projective plane.

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Example 1.1

Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{P} = \{a, b, c, d, e, f, g\}$, where

- $a = \{1, 2, 3\}$,
- $b = \{3, 4, 5\}$,
- $c = \{5, 6, 1\}$,
- $d = \{5, 7, 2\}$,
- $e = \{1, 7, 4\}$,
- $f = \{3, 7, 6\}$,
- $g = \{2, 4, 6\}$.

Then $(X, \mathcal{P})$ is a finite projective plane,[a] called the *Fano plane*.

[a] It is easy to check that (P1) and (P2) are satisfied. For (P0), we can take, for instance, $Q = \{1, 3, 5, 7\}$.

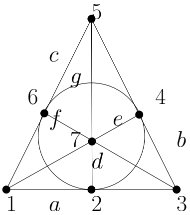


Figure: The Fano plane.

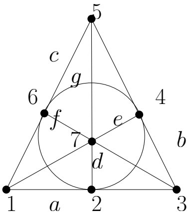
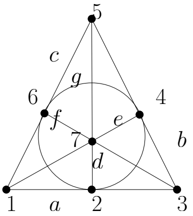


Figure: The Fano plane.

- In the picture above, the seven lines of the Fano plane are represented by six line segments and one circle.



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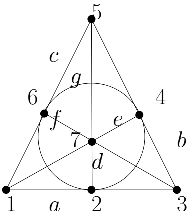

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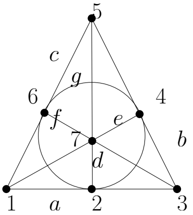
- In the picture above, the seven lines of the Fano plane are represented by six line segments and one circle.
- However, formally, each line of the Fano plane is simply a set of three points.
- Drawings can sometimes be useful for guiding our intuition. However, formal proofs should never rely on such pictures; instead, they should rely solely on the definition of a finite projective plane or on results (propositions, lemmas, theorems) proven about them.

- The *incidence graph* of a finite projective plane $(X, \mathcal{P})$ is a bipartite graph with bipartition $(X, \mathcal{P})$, in which $x \in X$ and $P \in \mathcal{P}$ are adjacent if and only if $x \in P$.



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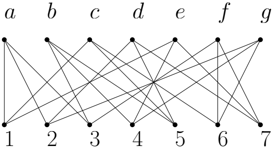
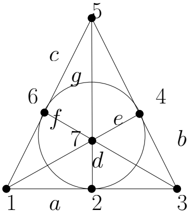


Figure: The incidence graph of the Fano plane.



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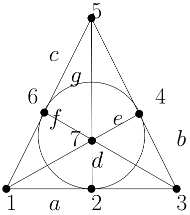

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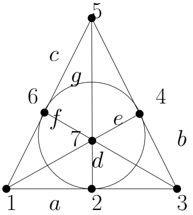

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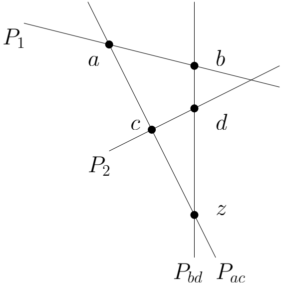

This proves the Claim.

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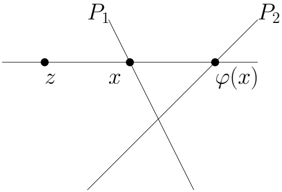

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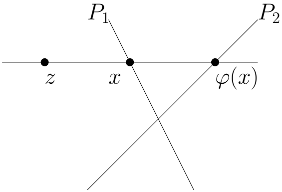

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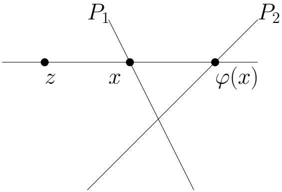

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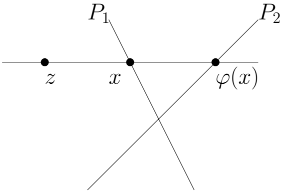

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[a] So, if $(X, \mathcal{P})$ is a finite projective plane of order n, then each line in $\mathcal{P}$ contains exactly $n + 1$ points.

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Proposition 1.3

The order of any finite projective plane is at least two.

Proof. Easy (read the Lecture Notes).

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Let $(X, \mathcal{P})$ be a finite projective plane of order n. Then all the following hold:

(a) for each point $x \in X$, exactly $n + 1$ lines in $\mathcal{P}$ pass through x;

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- We prove (c) after introducing "duality" (we use (a) and (b)).

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Proof (outline).

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Proof of the Claim (outline). Fix a point $x \in X$. Using (P0) from the definition of a finite projective plane, we fix a 4-element subset $Q \subseteq X$ s.t. for all $P \in \mathcal{P}$, $|Q \cap P| \leq 2$.

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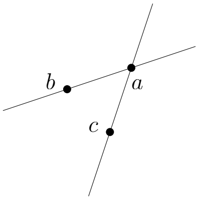
Claim. *For every point $x \in X$, there exists a line $P \in \mathcal{P}$ s.t. $x \notin P$.*

Proof of the Claim (outline, continued).



Then x belongs to at most one of $\overline{ab}$ and $\overline{ac}$. This proves the Claim.

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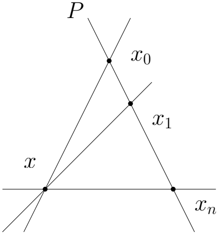
Proof of (a) (outline). Fix a point $x \in X$. By the Claim, there exists a line $P \in \mathcal{P}$ s.t. $x \notin P$. Since $(X, \mathcal{P})$ is of order n, we know that $|P| = n+1$; set $P = \{x_0, x_1, \dots, x_n\}$.



At least $n+1$ lines (namely, $\overline{xx_0}, \dots, \overline{xx_n}$) pass through x.

Every line through x intersects P, so $\overline{xx_0}, \dots, \overline{xx_n}$ are the only lines through x.

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For a set system $(X, \mathcal{S})$, we define the *dual* of $(X, \mathcal{S})$ to be the ordered pair $(Y, \mathcal{T})$, where $Y = \mathcal{S}$ and

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Example 2.1

Let $X = \{1, 2, 3\}$ and $\mathcal{S} = \{A, B\}$, where $A = \{1, 2\}$ and $B = \{1, 3\}$. Then the dual of $(X, \mathcal{S})$ is $(Y, \mathcal{T})$, where $Y = \{A, B\}$ and $\mathcal{T} = \Big\{ \{A, B\}, \{A\}, \{B\} \Big\}$.[a]

[a] Indeed $\{S \in \mathcal{S} \mid 1 \in S\} = \{A, B\}$, $\{S \in \mathcal{S} \mid 2 \in S\} = \{A\}$, and $\{S \in \mathcal{S} \mid 3 \in S\} = \{B\}$.

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- Then by definition, $(Y, \mathcal{R})$ is the dual of $(X, \mathcal{P})$.
- For all distinct $x_1, x_2 \in X$, there is a unique $P \in \mathcal{P}$ s.t. $x_1, x_2 \in P$.
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- A bit more work for (P0).

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