NDMI011: Combinatorics and Graph Theory 1

Lecture #4
Finite projective planes (part I)

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## 1 Finite projective planes: definition and basic properties

For a set $X$, the *power set* of $X$, denoted by $\mathscr{P}(X)$, is the set of all subsets of $X$. For example, if $X = \{1,2,3\}$, then

$$\mathscr{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Obviously, for any set $X$, we have that $\emptyset \in \mathscr{P}(X)$ and $X \in \mathscr{P}(X)$. Furthermore, if $X$ is finite, then $|\mathscr{P}(X)| = 2^{|X|}$.

A *set system* is an ordered pair $(X, \mathcal{S})$ such that $X$ is a set (called the *ground set*) and $\mathcal{S} \subseteq \mathscr{P}(X)$.

A *finite projective plane* is set system $(X, \mathcal{P})$ such that $X$ is a finite, and the following three properties are satisfied:

(P0) there exists a 4-element subset $Q \subseteq X$ such that every $P \in \mathcal{P}$ satisfies $|P \cap Q| \leq 2$;

(P1) all distinct $P_1, P_2 \in \mathcal{P}$ satisfy $|P_1 \cap P_2| = 1$;

(P2) for all distinct $x_1, x_2 \in X$, there exists a unique $P \in \mathcal{P}$ such that $x_1, x_2 \in P$.

If $(X, \mathcal{P})$ is a finite projective plane, then members of $X$ are called *points*, and members of $\mathcal{P}$ are called *lines*. For a point $x \in X$ and a line $P \in \mathcal{P}$ such that $x \in P$, we say that the line $P$ is *incident* with the point $x$, or that $P$ *contains* $x$, or that $P$ *passes through* $x$. For distinct points $a, b \in X$, we denote by $\overline{ab}$ the unique line in $\mathcal{P}$ that contains $a$ and $b$ (the existence and uniqueness of such a line follow from (P2)).

Finite projective planes (defined above) and the usual Euclidean planes (i.e. planes that you studied in high school) have some obvious similarities,

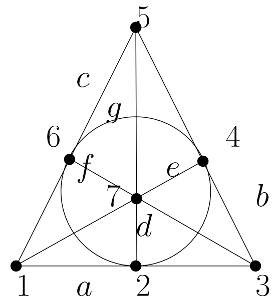


Figure 1.1: The Fano plane.

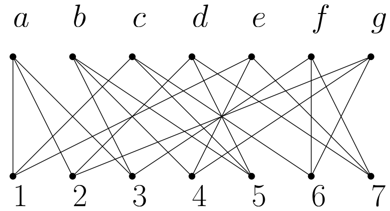


Figure 1.2: The incidence graph of the Fano plane.

but also some obvious differences. In a Euclidean plane, two distinct lines may intersect in at most one point, but distinct, parallel lines have an empty intersection. In finite projective planes, there are no "parallel lines": by (P1), two distinct lines always intersect in exactly one point, called their *point of intersection* or *intersection point*. Property (P2) from the definition of a finite projective plane is the same as for the Euclidean plane.

**Example 1.1.** *Let* $X = \{1, 2, 3, 4, 5, 6, 7\}$ *and* $\mathcal{P} = \{a, b, c, d, e, f, g\}$*, where*

- $a = \{1, 2, 3\}$,
- $b = \{3, 4, 5\}$,
- $c = \{5, 6, 1\}$,
- $d = \{5, 7, 2\}$,
- $e = \{1, 7, 4\}$,
- $f = \{3, 7, 6\}$,
- $g = \{2, 4, 6\}$.

*Then* $(X, \mathcal{P})$ *is a finite projective plane,*[1] *called the* Fano plane *(see Figure 1.1).*

Note that in Figure 1.1, the seven lines of the Fano plane are represented by six line segments and one circle. However, formally, each line of the Fano

[1] It is easy to check that (P1) and (P2) are satisfied. For (P0), we can take, for instance, $Q = \{1, 3, 5, 7\}$.

plane is simply a set of three points. Drawings such as the one in Figure 1.1 can sometimes be useful for guiding our intuition. However, formal proofs should never rely on such pictures; instead, they should rely solely on the definition of a finite projective plane or on results (propositions, lemmas, theorems) proven about them.[2]

To each finite projective plane $(X, \mathcal{P})$, we associate an "incidence graph" defined as follows. The *incidence graph* of a finite projective plane $(X, \mathcal{P})$ is a bipartite graph with bipartition $(X, \mathcal{P})$,[3] in which $x \in X$ and $P \in \mathcal{P}$ are adjacent if and only if $x \in P$. The incidence graph of the Fano plane is represented in Figure 1.2.

Note that each line of the Fano plane (see Example 1.1 and Figure 1.1) contains the same number of points. As our next proposition shows, this is not an accident.

**Proposition 1.2.** *Let $(X, \mathcal{P})$ be a finite projective plane. Then all lines in $\mathcal{P}$ have the same number of points.*

*Proof.* Fix $P_1, P_2 \in \mathcal{P}$. We must show that $|P_1| = |P_2|$.

> **Claim.** There exists a point $z \in X$ such that $z \notin P_1 \cup P_2$.

*Proof of the Claim.* First, using (P0) from the definition of a finite projective plane, we fix a 4-element subset $Q \subseteq X$ such that for all $P \in \mathcal{P}$, we have that $|Q \cap P| \leq 2$. If $Q \not\subseteq P_1 \cup P_2$, then we take any $z \in Q \setminus (P_1 \cup P_2)$, and we are done. So assume that $Q \subseteq P_1 \cup P_2$.

Since $|Q| = 4$ and $|Q \cap P_1|, |Q \cap P_2| \leq 2$, we now deduce that $Q \cap P_1$ and $Q \cap P_2$ are disjoint and each contain exactly two points. Set $Q \cap P_1 = \{a, b\}$ and $Q \cap P_2 = \{c, d\}$. We now consider the lines $P_{ac} := \overline{ac}$ and $P_{bd} := \overline{bd}$.[4] Since no line in $\mathcal{P}$ contains more than two points of $Q$, and since $a, c \in Q \cap P_{ac}$, we see that $Q \cap P_{ac} = \{a, c\}$. Similarly, $Q \cap P_{bd} = \{b, d\}$. Since $P_1, P_2, P_{ac}, P_{bd}$ have pairwise distinct intersections with the set $Q$, we see that the lines $P_1, P_2, P_{ac}, P_{bd}$ are pairwise distinct.

Now, by (P1), we have that $|P_{ac} \cap P_{bd}| = 1$; set $P_{ac} \cap P_{bd} = \{z\}$ (see the picture below).

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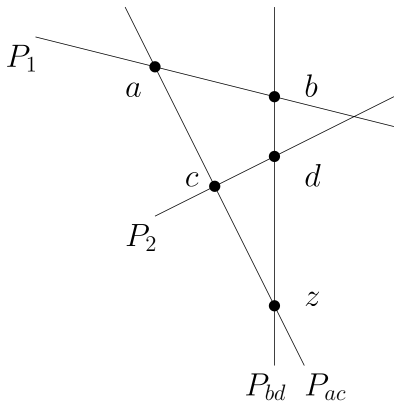
[2] The proofs of those results must, ultimately, rely only on the definition of a finite projective plane.

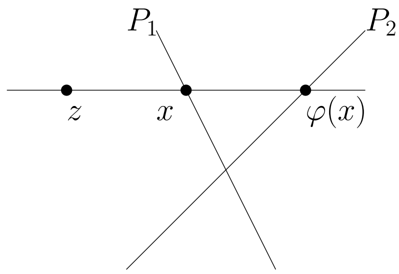
[3] So, in our incidence graph, $X$ and $\mathcal{P}$ are stable (i.e. independent) sets. (A *stable set*, also called an *independent set*, in a graph $G$ is any set of pairwise non-adjacent vertices of $G$.)

[4] Recall that, by (P2), there exists a unique line in $\mathcal{P}$ that contains both $a$ and $c$, and according to our notation, this line is denoted $\overline{ac}$. For convenience, we set $P_{ac} = \overline{ac}$. Similar remarks holds for $b, d$.




Since $P_{ac} \cap Q$ and $P_{bd} \cap Q$ are disjoint, we see that $z \notin Q$. If $z \in P_1$, then $a, z \in P_1 \cap P_{ac}$, which is impossible because $a, z$ are distinct points,[5] $P_1, P_{ac}$ are distinct lines, and by (P1), any two distinct lines intersect in exactly one point. Thus, $z \notin P_1$, and similarly, $z \notin P_2$. This proves the Claim. ■

Let $z$ be as in the Claim. We now define a function $\varphi : P_1 \to P_2$, as follows. For all $x \in P_1$, let $\varphi(x)$ be the unique point in the intersection of the lines $\overline{xz}$ and $P_2$ (see the picture below); by (P1) and (P2), our function $\varphi$ is well-defined.[6]




Let us check that $\varphi : P_1 \to P_2$ is surjective (i.e. onto). Fix $y \in P_2$, and let $x$ be the point of intersection of the lines $P_1$ and $yz$.[7] Then $y$ is the point of intersection of lines $\overline{xz}$ and $P_2$, and it follows that $y = \varphi(x)$. So, $\varphi : P_1 \to P_2$ is surjective. This implies that $|P_1| \geq |P_2|$. By symmetry, we also have that $|P_2| \geq |P_1|$, and we deduce that $|P_1| = |P_2|$. □

The *order* of a finite projective plane $(X, \mathcal{P})$ is the number $|P| - 1$, where

[5] We know that $a \neq z$ because $a \in Q$ and $z \notin Q$.

[6] Let us check that this in detail. First, since $z \notin P_1$, we know that for all $x \in P_1$, we have that $x \neq z$, and so by (P2), there is exactly one line (which we denoted by $\overline{xz}$) that passes through $x$ and $z$; furthermore, since $z \notin P_2$, we have that $\overline{xz}$ and $P_2$ are distinct lines, and so (P1) guarantees that $\overline{xz}$ and $P_2$ intersect in exactly one point, and we call this point $\varphi(x)$. Thus, $\varphi$ is well-defined. (We remark that it is possible that $P_1 = P_2$; in this case, the function $\varphi$ is simply the identity function on $P_1 = P_2$, that is, $\varphi(x) = x$ for all $x \in P_1$.)

[7] Check that $x$ exists and is unique!

$P$ is any line in $\mathcal{P}$.[8] By Proposition 1.2, this is well-defined. Note that the Fano plane has order two. Furthermore, the following proposition shows that the order of any finite projective plane is at least two.

**Proposition 1.3.** *The order of any finite projective plane is at least two.*

*Proof.* Let $(X, \mathcal{P})$ be a finite projective plane. It suffices to show that some line in $\mathcal{P}$ passes through at least three points. Using (P0) from the definition of a finite projective plane, we fix a 4-element subset $Q \subseteq X$ such that for all $P \in \mathcal{P}$, we have that $|Q \cap P| \leq 2$. Set $Q = \{a, b, c, d\}$. Consider the lines $P_{ab} := \overline{ab}$ and $P_{cd} := \overline{cd}$. Since $Q$ intersects each line in $\mathcal{P}$ in at most two points, we see that $Q \cap P_{ab} = \{a, b\}$ and $Q \cap P_{cd} = \{c, d\}$; in particular, $P_{ab} \neq P_{cd}$. By (P1), $P_{ab}$ and $P_{cd}$ intersect in exactly one point, call it $z$. Since $Q \cap P_{ab}$ and $Q \cap P_{cd}$ are disjoint, we see that $z \notin Q$. But now $P_{ab}$ contains at least three points, namely $a, b, z$. $\square$

**Theorem 1.4.** *Let $(X, \mathcal{P})$ be a finite projective plane of order $n$.[9] Then all the following hold:*

*(a) for each point $x \in X$, exactly $n + 1$ lines in $\mathcal{P}$ pass through $x$;*

*(b) $|X| = n^2 + n + 1$;*

*(c) $|\mathcal{P}| = n^2 + n + 1$.*

*Proof of (a) and (b).* We begin by proving an auxiliary claim.

> **Claim.** For every point $x \in X$, there exists a line $P \in \mathcal{P}$ such that $x \notin P$.

*Proof of the Claim.* Fix a point $x \in X$. Using (P0) from the definition of a finite projective plane, we fix a 4-element subset $Q \subseteq X$ such that for all $P \in \mathcal{P}$, we have that $|Q \cap P| \leq 2$. Clearly, $|Q \setminus \{x\}| \geq 3$. Let $a, b, c \in Q \setminus \{x\}$ be pairwise distinct. It now suffices to show that $x$ belongs to at most one of $\overline{ab}$ and $\overline{ac}$. Suppose otherwise, i.e. suppose that $x$ belongs both to $\overline{ab}$ and to $\overline{ac}$. Then the lines $\overline{ab}$ and $\overline{ac}$ have at least two points (namely, $a$ and $x$) in common, and so by (P2), we have that $\overline{ab} = \overline{ac}$. But now the line $\overline{ab} = \overline{ac}$ contains at least three points (namely, $a, b, c$) of $Q$, a contradiction. This proves the Claim. $\blacksquare$

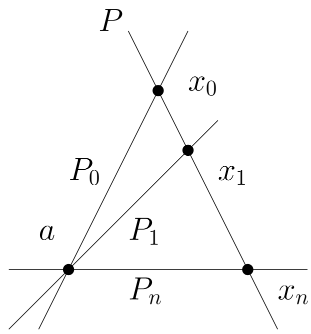
We now prove (a). Fix a point $x \in X$. By the Claim, there exists a line $P \in \mathcal{P}$ such that $x \notin P$. Since $(X, \mathcal{P})$ is of order $n$, we know that $|P| = n+1$; set $P = \{x_0, x_1, \dots, x_n\}$. By (P2), the lines $\overline{xx_0}, \overline{xx_1}, \dots, \overline{xx_n}$ are pairwise

[8] So, if $(X, \mathcal{P})$ is a finite projective plane of order $n$, then each line in $\mathcal{P}$ contains exactly $n + 1$ points.

[9] By Proposition 1.3, we have that $n \geq 2$.

distinct,[10] and they all contain $x$. So, there are at least $n+1$ lines passing through $x$. On the other hand, by (P1), any line passing though $x$ intersects the line $P$ in one of the points $x_0, x_1, \dots, x_n$, and is therefore (by (P2)) equal to one of $\overline{xx_0}, \dots, \overline{xx_1}, \overline{xx_n}$. Thus, exactly $n+1$ lines pass through $x$. This proves (a).

We now prove (b). Fix any line $P \in \mathcal{P}$. Since $(X, \mathcal{P})$ is of order $n$, we know that $|P| = n+1$; set $P = \{x_0, x_1, \dots, x_n\}$. Since every line in $\mathcal{P}$ has $n+1$ points, the Claim guarantees that $|X| \geq n+2$; consequently, $P \subsetneqq X$. Fix any $a \in X \setminus P$. For each $i \in \{0, 1, \dots, n\}$, we set $P_i := \overline{ax_i}$ (see the picture below).




By (P2), lines $P_0, P_1, \dots, P_n$ are pairwise distinct,[11] and so by (P1), any two of them have exactly one point in common. Since $a$ lies on each of $P_0, P_1, \dots, P_n$, we see that $P_i \cap P_j = \{a\}$ for all distinct $i, j \in \{0, 1, \dots, n\}$; consequently, $P_0 \setminus \{a\}, P_1 \setminus \{a\}, \dots, P_n \setminus \{a\}$ are pairwise disjoint. Now, since $(X, \mathcal{P})$ is of order $n$, we know that $P_0, P_1, \dots, P_n$ each have $n+1$ points, and we deduce that

$$\begin{aligned} |P_0 \cup P_1 \cup \dots \cup P_n| &= |\{a\}| + |P_0 \setminus \{a\}| + |P_1 \setminus \{a\}| + \dots + |P_n \setminus \{a\}| \\ &= 1 + (n+1)n \\ &= n^2 + n + 1. \end{aligned}$$

It now remains to show that $X = P_0 \cup P_1 \cup \dots \cup P_n$; in fact, we only need to show that $X \subseteq P_0 \cup P_1 \cup \dots \cup P_n$, for the reverse inclusion is immediate. Fix a point $x \in X$; we must show that $x$ belongs to at least one of $P_0, P_1, \dots, P_n$. We may assume that $x \neq a$, for otherwise we are done. The line $R := \overline{xa}$ is distinct from $P$ (because $a \in R$, but $a \notin P$), and so by (P1), $|P \cap R| = 1$. Since $P = \{x_0, x_1, \dots, x_n\}$, it follows that there exists some $i \in \{0, 1, \dots, n\}$

[10] Indeed, suppose that for some distinct $i, j \in \{0, 1, \dots, n\}$, we had $\overline{xx_i} = \overline{xx_j}$. Now the line $\overline{xx_i} = \overline{xx_j}$ contains both $x_i$ and $x_j$. On the other hand, we know that the line $P$ contains both $x_i$ and $x_j$. By (P2), there is exactly one line that contains both $x_i$ and $x_j$, and we deduce that $P = \overline{xx_i} = \overline{xx_j}$. But this implies that $x \in P$, contrary to the choice of $P$.

[11] This is analogous to the argument from footnote 10.

such that $P \cap R = \{x_i\}$. Now lines $P_i$ and $R$ have at least two points (namely, $a$ and $x_i$) in common, and so by (P2), we have that $R = P_i$. Since $x \in R$, we deduce that $x \in P_i$. This completes the argument. $\square$

We postpone the proof of Theorem 1.4(c) to the end of section 2.

## 2 Duality

In this section, we show (roughly speaking) that by swapping the roles of points and lines of a finite projective plane, we obtain another finite projective plane (called the "dual" of the original finite projective plane).

Let us be more precise. For a set system $(X, \mathcal{S})$, we define the *dual* of $(X, \mathcal{S})$ to be the ordered pair $(Y, \mathcal{T})$, where $Y = \mathcal{S}$ and

$$\mathcal{T} = \Big\{ \{S \in \mathcal{S} \mid x \in S\} \ \mid \ x \in X \Big\}.$$

**Example 2.1.** *Let $X = \{1, 2, 3\}$ and $\mathcal{S} = \{A, B\}$, where $A = \{1, 2\}$ and $B = \{1, 3\}$. Then the dual of $(X, \mathcal{S})$ is $(Y, \mathcal{T})$, where $Y = \{A, B\}$ and $\mathcal{T} = \Big\{ \{A, B\}, \{A\}, \{B\} \Big\}$.*[12]

Theorem 2.2 (below) states that the dual of a finite projective plane is again a finite projective plane. Before giving a formal proof, let us try to give some intuition behind this. If $(X, \mathcal{P})$ is a finite projective plane, and $(Y, \mathcal{R})$ is its dual, then the lines of $(X, \mathcal{P})$ become points of $(Y, \mathcal{R})$ (indeed, by definition, $Y = \mathcal{P}$). Furthermore, points of $(X, \mathcal{P})$ correspond to the lines of $(Y, \mathcal{R})$ in a natural way: a point $x \in X$ corresponds to the line $R_x := \{P \in \mathcal{P} \mid x \in P\} \in \mathcal{R}$. The incidence graphs of $(X, \mathcal{P})$ and $(Y, \mathcal{R})$ are isomoprhic (i.e. identical up to a relabeling of the vertices), except that points turn into lines and vice versa.

**Theorem 2.2.** *The dual of a finite projective plane is again a finite projective plane.*

*Proof.* Let $(X, \mathcal{P})$ be a finite projective plane, and let $(Y, \mathcal{R})$ be its dual. To simplify notation, for all $x \in X$, we set $R_x = \{P \in \mathcal{P} \mid x \in P\}$. We now have that $Y = \mathcal{P}$ and $\mathcal{R} = \{R_x \mid x \in X\}$.[13] Obviously, for all $x \in X$, we have that $R_x \subseteq \mathcal{P} = Y$, and consequently $R_x \in \mathscr{P}(Y)$; thus, $\mathcal{R} \subseteq \mathscr{P}(Y)$, i.e. $(Y, \mathcal{R})$ is a set system. Furthermore, since $X$ is finite, and since $Y = \mathcal{P} \subseteq \mathcal{P}(X)$, we have that $Y$ is finite. It now remains to show that $(Y, \mathscr{R})$ satisfies (P0), (P1), and (P2).

We first prove that $(Y, \mathcal{R})$ satisfies (P0). Since $(X, \mathcal{P})$ is a finite projective plane, (P0) guarantees that there exists a 4-element set $Q \subseteq X$ such that

[12] Indeed $\{S \in \mathcal{S} \mid 1 \in S\} = \{A, B\}$, $\{S \in \mathcal{S} \mid 2 \in S\} = \{A\}$, and $\{S \in \mathcal{S} \mid 3 \in S\} = \{B\}$.

[13] It is not hard to show (details?) that for all $x_1, x_2 \in X$, we have that $R_{x_1} = R_{x_2}$ if and only if $x_1 = x_2$. However, we will not use this fact.

for all $P \in \mathcal{P}$, we have that $|Q \cap P| \leq 2$. Set $Q = \{a, b, c, d\}$. Further, set $P_1 = \overline{ab}$, $P_2 = \overline{bc}$, $P_3 = \overline{cd}$, and $P_4 = \overline{da}$. Since $|Q \cap P| \leq 2$ for all $P \in \mathcal{P}$, we now deduce that $Q \cap P_1 = \{a, b\}$, $Q \cap P_2 = \{b, c\}$, $Q \cap P_3 = \{c, d\}$, and $Q \cap P_4 = \{d, a\}$; in particular, every point of $Q$ belongs to exactly two of $P_1, P_2, P_3, P_4$. Now, set $Q^* = \{P_1, P_2, P_3, P_4\}$; we must show that no element of $\mathcal{R}$ contains more than two elements of $Q^*$. Suppose otherwise. Then there exist some $x \in X$ and pairwise distinct $i, j, k \in \{1, 2, 3\}$ such that $P_i, P_j, P_k \in R_x$; consequently, $x \in P_i \cap P_j \cap P_k$. Since each point in $Q$ belongs to exactly two of $P_1, P_2, P_3, P_4$, whereas $x$ belongs to at least three of them, we see that $x \notin Q$. On the other hand, for any three of $P_1, P_2, P_3, P_4$, some two of them have a point of $Q$ in common. So, some two of $P_i, P_j, P_k$, have at least two points in common (namely, one point of $Q$, plus the point $x$) and are therefore (by (P2) applied to $(X, \mathcal{P})$) identical, a contradiction. This proves that $(Y, \mathcal{R})$ satisfies (P0).

We next show that $(Y, \mathcal{R})$ satisfies (P1). Fix distinct $R_1, R_2 \in \mathcal{R}$; we must show that $|R_1 \cap R_2| = 1$. By the construction of $\mathcal{R}$, there exist some $x_1, x_2 \in X$ such that $R_1 = R_{x_1}$ and $R_2 = R_{x_2}$; since $R_1 \neq R_2$, we have that $x_1 \neq x_2$. Now, $R_1 \cap R_2 = \{P \in \mathcal{P} \mid x_1, x_2 \in P\}$. By (P2) for $(X, \mathcal{P})$, there is exactly one $P \in \mathcal{P}$ such that $x_1, x_2 \in P$; so, $R_1 \cap R_2 = \{P\}$, and in particular, $|R_1 \cap R_2| = 1$. Thus, $(Y, \mathcal{R})$ satisfies (P1).

It remains to show that $(Y, \mathcal{R})$ satisfies (P2). Fix distinct $P_1, P_2 \in Y (= \mathcal{P})$; we must show that there is exactly one member of $\mathcal{R}$ that contains both $P_1$ and $P_2$. By (P1) for $(X, \mathcal{P})$, we know that $|P_1 \cap P_2| = 1$; set $P_1 \cap P_2 = \{x_0\}$. So, $R_{x_0}$ is the only member of $\mathcal{R}$ that contains both $P_1$ and $P_2$. Thus, $(Y, \mathcal{R})$ satisfies (P2). $\square$

**Notation:** The dual of a finite projective plane $(X, \mathcal{P})$ is sometimes denoted by $(X, \mathcal{P})^*$.

We complete this section by proving Theorem 1.4(c), as follows. Let $(X, \mathcal{P})$ be a finite projective plane of order $n$; we must show that $|\mathcal{P}| = n^2 + n + 1$. By Theorem 2.2, $(Y, \mathcal{R}) := (X, \mathcal{P})^*$ is also a finite projective plane. By Theorem 1.4(a), we have that for all $x \in X$, there are exactly $n+1$ lines $P \in \mathcal{P}$ that contain $x$. It then follows from the construction that all $R \in \mathcal{R}$ satisfy $|R| = n + 1$.[14] So, the finite projective plane $(Y, \mathcal{R})$ is of order $n$. By Theorem 1.4(b), we now have that $|Y| = n^2 + n + 1$. But $Y = \mathcal{P}$, and so $|\mathcal{P}| = n^2 + n + 1$, which is what we needed to show.

[14] Let us check this. First, for all $x \in X$, we set $R_x = \{P \in \mathcal{P} \mid x \in P\}$, as in the proof of Theorem 2.2. Since every point in $X$ belongs to precisely $n + 1$ lines in $\mathcal{P}$, we see that for all $x \in X$, we have that $|R_x| = n + 1$. Since $\mathcal{R} = \{R_x \mid x \in X\}$, we deduce that all members of $\mathcal{R}$ have precisely $n + 1$ elements.