NDMI011: Combinatorics and Graph Theory 1

Lecture #3
Generating functions (part II)

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1 Basic operations with generating functions

We begin by recalling the definition of a generating function. Suppose $\{a_n\}_{n=0}^{\infty}$ is some infinite sequence of real (or complex) numbers. The *generating function* of this sequence is the power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

We now consider some ways of combining generating functions. Suppose $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences of real (or complex) numbers, and suppose $a(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $b(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ are the corresponding generating functions. Further, suppose α is a real (or complex) constant. Then we have the following.

1. The generating function of the sequence $\{a_n + b_n\}_{n=0}^{\infty}$ is $a(x) + b(x)$.

2. The generating function of the sequence $\{a_n - b_n\}_{n=0}^{\infty}$ is $a(x) - b(x)$.

3. The generating function of the sequence $\{\alpha a_n\}_{n=0}^{\infty}$ is $\alpha a(x)$.

4. For an integer $k \geq 1$, the generating function of the sequence

   $$\underbrace{0, \dots, 0}_{k}, a_0, a_1, a_2, \dots$$

   is $x^k a(x)$.

5. For an integer $k \geq 1$, the generating function of the sequence $\{a_{n+k}\}_{n=0}^{\infty}$, i.e. the sequence $a_k, a_{k+1}, a_{k+2}, \ldots$, is $\frac{1}{x^k}\Big(a(x) - \sum_{i=0}^{k-1} a_i x^i\Big)$.[1]

6. The generating function of the sequence $\{\alpha^n a_n\}_{n=0}^{\infty}$ is $c(x) = a(\alpha x)$.[2]

7. For an integer $k \geq 1$, the generating function of the sequence

$$a_0, \underbrace{0, \ldots, 0}_{k}, a_1, \underbrace{0, \ldots, 0}_{k}, a_2, \underbrace{0, \ldots, 0}_{k}, a_3, \ldots$$

is $a(x^{k+1})$.[3]

8. The generating function of the sequence $\{(n+1)a_{n+1}\}_{n=0}^{\infty}$, i.e. the sequence $a_1, 2a_2, 3a_3, 4a_4, \ldots$, is $a'(x)$. The generating function for the sequence $0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \frac{1}{4}a_3, \ldots$ is $\int_0^x a(t)dt$. (We differentiate and integrate power series term-by-term.)

9. The function $c(x) = a(x)b(x)$ is the generating function of the sequence $\{c_n\}_{n=0}^{\infty}$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for each integer $n \geq 0$.[4]

Example 1.1. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence, and let $a(x)$ be its generating function. Find the generating function of the sequence $a_0, 0, a_2, 0, a_4, \ldots$ in terms of the function $a(x)$.*

Solution. We observe that $a_0, 0, a_2, 0, a_4, \ldots$ is the sum of the following two sequences: $\{\frac{a_n}{2}\}_{n=0}^{\infty}$ and $\{\frac{(-1)^n a_n}{2}\}_{n=0}^{\infty}$. The generating function of $\{\frac{a_n}{2}\}_{n=0}^{\infty}$ is $\frac{1}{2}a(x)$, and the generating function of $\{\frac{(-1)^n a_n}{2}\}_{n=0}^{\infty}$ is $\frac{1}{2}a(-x)$. So, the generating function of $a_0, 0, a_2, 0, a_4, \ldots$ is $\frac{a(x)+a(-x)}{2}$. □

Example 1.2. *Find (the closed form of) the generating function of the sequence $1, 1, 2, 2, 4, 4, 8, 8, 16, 16, \ldots$, i.e. the sequence $\{2^{\lfloor n/2 \rfloor}\}_{n=0}^{\infty}$.*

Solution. Recall that the generating function of the sequence $1, 2, 4, 8, 16, \ldots$ is $\frac{1}{1-2x}$. The generating function of $1, 0, 2, 0, 4, 0, 8, 0, \ldots$ is $\frac{1}{1-2x^2}$, and the generating function of $0, 1, 0, 2, 0, 4, 0, 8, 0, \ldots$ is $\frac{x}{1-2x^2}$. So, the generating function of $1, 1, 2, 2, 4, 4, 8, 8, 16, 16, \ldots$ is the sum of these two functions, i.e. $\frac{1+x}{1-2x^2}$. □

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[1] For example, the generating function of the sequence $a_3, a_4, a_5, \ldots$ is

$$\frac{1}{x^3}\Big(a(x) - (a_0 + a_1 x + a_2 x^2)\Big).$$

[2] For instance, since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is the generating function of $1, 1, 1, 1, 1, \ldots$, we see that $\frac{1}{1-2x}$ is the generating function of $1, 2, 4, 8, 16, \ldots$.

[3] For instance, the generating function of the sequence $a_0, 0, 0, a_1, 0, 0, a_2, 0, 0, a_3, \ldots$ is $a(x^3)$.

[4] So, $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b_0$, $c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$, etc.

Example 1.3. *Find (the closed form of) the generating function of the sequence $1^2, 2^2, 3^2, 4^2, \dots$, i.e. the sequence $\{(n+1)^2\}_{n=0}^{\infty}$.*

Solution. The generating function of the sequence $1, 1, 1, 1, \dots$ is $\frac{1}{1-x}$. By differentiating, we see that $\frac{d}{dx}(\frac{1}{1-x}) = \frac{1}{(1-x)^2}$ is the generating function of the sequence $1, 2, 3, 4, \dots$, i.e. the sequence $\{n+1\}_{n=0}^{\infty}$. By differentiating again, we see that $\frac{d}{dx}(\frac{1}{(1-x)^2}) = \frac{2}{(1-x)^3}$ is the generating sequence of the sequence $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, \dots$, i.e. the sequence $\{(n+1)(n+2)\}_{n=0}^{\infty}$. Now, $(n+1)^2 = (n+1)(n+2) - (n+1)$ for all integers $n \geq 0$, and we have computed the generating functions for the sequences $\{(n+1)(n+2)\}_{n=0}^{\infty}$ and $\{n+1\}_{n=0}^{\infty}$. So, the generating function of $\{(n+1)\}_{n=0}^{\infty}$ is

$$a(x) = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}.$$

□

2 An application of generating functions: counting binary trees

In this section, we consider binary trees of the sort that are often used in data structures. For our purposes, we can define binary trees recursively as follows: a *binary tree* is either empty (i.e. contains no nodes), or consists of designated node r (called the *root*), plus an ordered pair (T_L, T_R) of binary trees, where T_L and T_R (called the *left subtree* and the *right subtree*) have disjoint sets of nodes and do not contain the node r (see Figure 2.1). The empty binary tree has zero nodes, and if a binary tree T consists of a root r and an ordered pair (T_L, T_R) of binary trees, then the number of nodes of T is $1 + n_L + n_R$, where n_L is the number of nodes of T_L, and n_R is the number of nodes of T_R.

For each integer $n \geq 0$, let b_n be the number of binary trees on n nodes, and let $b(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ be the generating function of the sequence $\{b_n\}_{n=0}^{\infty}$. It is easy to check that $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, and $b_3 = 5$ (see Figure 2.2). Now, let us find a recursive formula for b_n ($n \geq 1$). The number of binary trees on $n \geq 1$ nodes is equal to the number of ordered pairs (T_L, T_R) of binary trees such that T_L, T_R together have $n-1$ nodes. Thus, for all integers $n \geq 1$, we have that

$$b_n \quad = \quad b_0 b_{n-1} + b_1 b_{n-2} + \dots + b_{n-1} b_0 \quad = \quad \sum_{k=0}^{n-1} b_k b_{n-k-1}.$$

Since $b_0 = 1$, this implies that

$$b(x) = 1 + x b(x)^2.$$

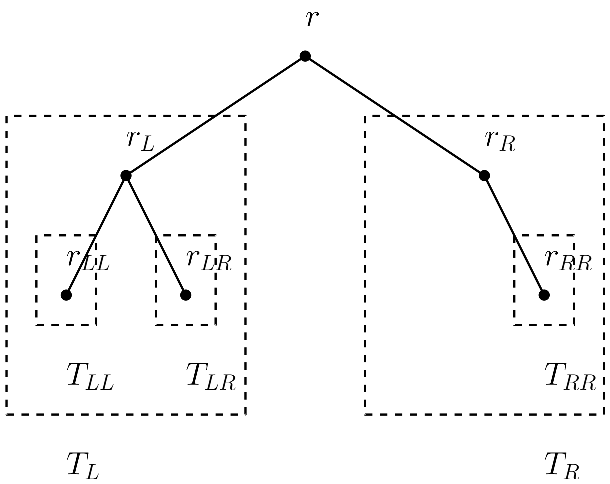

Figure 2.1: A binary tree on six nodes. Note that the left subtree of the right subtree is empty.

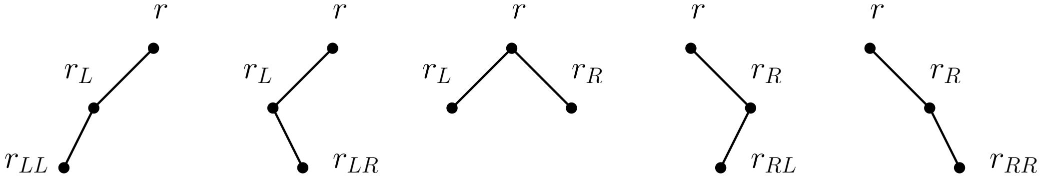

Figure 2.2: All the binary trees on three nodes.

Using the quadratic formula,[5] we get that either

$$b(x) = \tfrac{1-\sqrt{1-4x}}{2x} \qquad \text{or} \qquad b(x) = \tfrac{1+\sqrt{1-4x}}{2x}.$$

We must now determine which of these two formulas is the correct one.

Since $b_0 = 1$, we have that $\lim\limits_{x\to 0^+} b(x) = 1$. Since

$$\begin{aligned}
\lim_{x\to 0^+} \tfrac{1-\sqrt{1-4x}}{2x} &= \lim_{x\to 0^+} \left(\tfrac{1-\sqrt{1-4x}}{2x} \cdot \tfrac{1+\sqrt{1-4x}}{1+\sqrt{1-4x}}\right) \\
&= \lim_{x\to 0^+} \tfrac{1-(1-4x)}{2x(1+\sqrt{1-4x})} \\
&= \lim_{x\to 0^+} \tfrac{2}{1+\sqrt{1-4x}} \\
&= 1,
\end{aligned}$$

whereas

$$\lim_{x\to 0^+} \tfrac{1+\sqrt{1-4x}}{2x} = \infty,$$

we deduce that[6]

$$b(x) = \tfrac{1-\sqrt{1-4x}}{2x}.$$

By the Generalized Binomial Theorem, we have that

$$\begin{aligned}
\sqrt{1-4x} &= \sum_{n=0}^{\infty} \tbinom{1/2}{n}(-4x)^n \\
&= \sum_{n=0}^{\infty} (-4)^n \tbinom{1/2}{n} x^n \\
&= 1 + \sum_{n=1}^{\infty} (-4)^n \tbinom{1/2}{n} x^n \qquad \text{because } (-4)^0 \tbinom{-1/2}{0} x^0 = 1 \\
&= 1 + x\sum_{n=0}^{\infty} (-4)^{n+1} \tbinom{1/2}{n+1} x^n
\end{aligned}$$

and consequently,

$$1 - \sqrt{1-4x} = -x\sum_{n=0}^{\infty} (-4)^{n+1} \tbinom{1/2}{n+1} x^n.$$

[5]Here, we treat $b(x)$ as the variable and x as a constant.

[6]Since we have x in the denominator, $b(0)$ is not defined, and in particular, $b(x)$ does not have a Maclaurin series. However, as we shall see, the constant term in the Maclaurin expansion of $1-\sqrt{1-4x}$ is zero, and so we can simply divide the resulting series by $2x$ and thus obtain another series, which is precisely the generating function (expressed as a power series) of the sequence $\{b_n\}_{n=0}^{\infty}$.

It follows that

$$b(x) \quad = \quad \tfrac{1-\sqrt{1-4x}}{2x} \quad = \quad \sum_{n=0}^{\infty} (-\tfrac{1}{2})(-4)^{n+1}\tbinom{1/2}{n+1}x^n.$$

Thus, for all non-negative integers n, we have that

$$b_n \quad = \quad (-\tfrac{1}{2})(-4)^{n+1}\tbinom{1/2}{n+1}.$$

Let us now try to obtain a nicer formula for b_n. For an integer $n \geq 0$, we compute:

$$\begin{aligned}
b_n &= (-\tfrac{1}{2})(-4)^{n+1}\tbinom{1/2}{n+1} \\
&= (-\tfrac{1}{2})(-4)^{n+1}\tfrac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-n)}{(n+1)!} \\
&= (-\tfrac{1}{2})(-4)^{n+1}\tfrac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2n-1}{2})}{(n+1)!} \\
&= (-\tfrac{1}{2})(-4)^{n+1}(-1)^n(\tfrac{1}{2})^{n+1}\tfrac{1\cdot 3\cdot\dots\cdot(2n-1)}{(n+1)!} \\
&= 2^n \cdot \tfrac{1\cdot 3\cdot\dots\cdot(2n-1)}{(n+1)!} \\
&= \tfrac{2\cdot 4\cdot\dots\cdot(2n)}{n!} \cdot \tfrac{1\cdot 3\cdot\dots\cdot(2n-1)}{(n+1)!} \\
&= \tfrac{(2n)!}{n!(n+1)!} \\
&= \tfrac{1}{n+1}\tbinom{2n}{n}.
\end{aligned}$$

So, we have obtained that, for all integers $n \geq 0$, the number of binary trees on n nodes is

$$b_n \quad = \quad \tfrac{1}{n+1}\tbinom{2n}{n}.$$

We remark that the numbers $\frac{1}{n+1}\binom{2n}{n}$ above have a special name: they are called *Catalan numbers*.

3 An application of generating functions: random walks

We consider the following infinite random walk on the integer line $\mathbb{Z}$: we begin our walk at 1, and at each step, we move at random either two units to the right ($+2$) or one unit to the left (-1). We would like to determine the probability that we reach the origin at some point in our walk.

We proceed as follows. For each integer $n \geq 0$, let P_n be the probability that we reach the origin after at most n steps. Obviously, $\{P_n\}_{n=0}^{\infty}$ is a non-decreasing sequence, and it is bounded above by 1. So, by the Monotone Sequence Theorem, it converges. Let

$$P = \lim_{n\to\infty} P_n.$$

Then P is the probability that we need to compute.

Now, for each integer $n \geq 0$, let a_n be the number of n-step walks in which we reach the origin for the first time after precisely n steps.[7] Furthermore, the total number of n-step walks is 2^n. It follows that

$$P_n = \sum_{i=0}^{n} \frac{a_i}{2^i}$$

for all non-negative integers n, and consequently,

$$P = \sum_{n=0}^{\infty} \frac{a_n}{2^n}.$$

Let $a(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$. Note that

$$P = a\Big(\frac{1}{2}\Big).$$

For our solution, it will be useful to consider random walks that start at points other than 1 (but proceed according to the same rules: at each step, we move at random either two units to the right or one unit to the left). For an integer $n \geq 0$, let b_n be the number n-step of random walks (following our rules) starting at 2 and ending at the origin, without reaching the origin at any point during the walk (except at the very end). In such a walk, we cannot reach the origin without first reaching 1, and then reaching the origin from there. So, if we are to reach the origin for the first time after precisely n steps, starting at 2, then there must be some $k \in \{1, \ldots, n-1\}$ such that we reach 1 for the first time after precisely k steps,[8] and then starting at 1, we reach the origin for the first time after $n-k$ steps.[9] For each choice of $k \in \{1, \ldots, n-1\}$, there are $a_k a_{n-k}$ such walks, and so

$$\begin{aligned} b_n &= \sum_{k=1}^{n-1} a_k a_{n-k} \\ &= \sum_{k=0}^{n} a_k a_{n-k} \qquad \text{because } a_0 = 0 \end{aligned}$$

[7] So, at the end of our n-step walk, we are at the origin, and furthermore, we were not at the origin after k steps for any non-negative integer $k < n$.

[8] There are precisely a_k many k-step walks that have this property: the number of ways to reach 1 from 2 for the first time after k steps is the same as the number of ways to reach the origin from 1 for the first time after k steps.

[9] There are precisely a_{n-k} many $(n-k)$-step walks that have this property.

Now, if $b(x) = \sum_{n=0}^{\infty} b_n$ is the generating function for the sequence $\{b_n\}_{n=0}^{\infty}$, then we get that

$$b(x) = a(x)^2.$$

Next, we consider random walks starting at 3, and moving according to our rules. For each integer $n \geq 0$, let c_n be the number of n-step random walks (following our rules) starting at 3 and ending at the origin, without reaching the origin at any point during the walk (except at the very end). We now argue similarly to the above. In such a walk, we cannot reach the origin before first reaching 2, and then reaching the origin from there. So, if we are to reach the origin for the first time after precisely n steps, starting at 3, then there must be some $k \in \{1, \ldots, n-1\}$ such that we reach 2 for the first time after precisely k steps,[10] and then starting at 2, we reach the origin for the first time after $n-k$ steps.[11] For each choice of $k \in \{1, \ldots, n-1\}$, there are $a_k b_{n-k}$ such walks, and so

$$\begin{aligned} b_n &= \sum_{k=1}^{n-1} a_k b_{n-k} \\ &= \sum_{k=0}^{n} a_k b_{n-k} \qquad \text{because } a_0 = 0 \text{ and } b_0 = 0 \end{aligned}$$

Now, if $c(x) = \sum_{n=0}^{\infty} c_n$ is the generating function for the sequence $\{c_n\}_{n=0}^{\infty}$, then we get that $c(x) = a(x)b(x)$. We already saw that $b(x) = a(x)^2$, and so it follows that

$$c(x) = a(x)^3.$$

We now observe the following. Obviously, $a_0 = 0$ and $a_1 = 1$. Next, if we start at 1, then for an integer $n \geq 2$, there are precisely c_{n-1} ways to reach the origin for the first time after precisely n steps: we must first move two units to the right,[12] and then reach the origin from 3 for the first time after precisely $n-1$ steps. Thus, $a_n = c_{n-1}$ for all integers $n \geq 2$. We now

[10] There are precisely a_k many k-step walks that have this property.

[11] There are precisely b_{n-k} many $(n-k)$-step walks that have this property.

[12] Indeed, if we moved one unit to the left instead, then we would reach the origin after precisely one step, and so (since $n \geq 2$) we would not reach the origin for the first time after n steps.

compute:

$$\begin{aligned}
a(x) &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= x + x \sum_{n=2}^{\infty} a_n x^{n-1} && \text{because } a_0 = 0 \text{ and } a_1 = 1 \\
&= x + x \sum_{n=2}^{\infty} c_{n-1} x^{n-1} && \text{because } a_n = c_{n-1} \text{ for } n \geq 2 \\
&= x + x \sum_{n=1}^{\infty} c_n x^n \\
&= x + x \sum_{n=0}^{\infty} c_n x^n && \text{because } c_0 = 0 \text{ (obvious)} \\
&= x + xc(x).
\end{aligned}$$

So, we have obtained the equation $a(x) = x + xc(x)$, and we know from before that $c(x) = a(x)^3$. So, we have that

$$a(x) = x + xa(x)^3.$$

At this point, we could in principle use the cubic equation to compute $a(x)$,[13] and then compute $P = a(\frac{1}{2})$ by plugging in $x = \frac{1}{2}$ into the function a. However, there is a quicker and easier way. Since $P = a(\frac{1}{2})$, we have that

$$P = \frac{1}{2} + \frac{1}{2}P^3.$$

The equation above has three solutions: 1, $\frac{-1+\sqrt{5}}{2}$, and $\frac{-1-\sqrt{5}}{2}$.[14] Obviously, $P \geq 0$, and so $P \neq \frac{-1-\sqrt{5}}{2}$. To simplify notation, we set

$$\varphi = \frac{-1+\sqrt{5}}{2}.$$

(So, φ is the golden ratio.) We now have that either $P = 1$ or $P = \varphi$.

Let us show that $P \neq 1$, i.e. that $a(\frac{1}{2}) \neq 1$. First, $a(x) = \sum_{n=0}^{\infty} a_n x^n$ has non-negative coefficients and converges for $x = \frac{1}{2}$. So, the function a is

[13] But note that we would get three solutions, and we would have to figure out which one is the correct one.

[14] The equation $P = \frac{1}{2} + \frac{1}{2}P^3$ is equivalent to the equation $P^3 - 2P + 1 = 0$. Obviously, 1 is a root of the latter equation. We find the other two roots by first factoring $P^3 - 2P + 1 = (P-1)(P^2 + P - 1)$, and then using the quadratic equation to find the other two roots.

continuous and increasing on the interval $[0, \frac{1}{2}]$.[15] Obviously, $a(0) = a_0 = 0$. Suppose that $a(\frac{1}{2}) = 1$. Since $0 < \varphi < 1$, and since a is continuous on $[0, \frac{1}{2}]$, the Intermediate Value Theorem guarantees that there exists some $x_0 \in (0, \frac{1}{2})$ such that $a(x_0) = \varphi$. Since φ is a root of the equation $P = \frac{1}{2} + \frac{1}{2}P^3$, we have that

$$\varphi = \frac{1}{2} + \frac{1}{2}\varphi^3.$$

On the other hand, we know that $a(x_0) = x_0 + x_0 a(x_0)^3$, and so

$$\varphi = x_0 + x_0\varphi^3.$$

It follows that

$$\frac{1}{2} + \frac{1}{2}\varphi^3 = x_0 + x_0\varphi^3,$$

which implies that

$$(x_0 - \frac{1}{2})(\varphi^3 + 1) = 0,$$

which is false since $x_0 \neq \frac{1}{2}$ and $\varphi^3 \neq -1$. This proves that $P \neq 1$, and it follows that $P = \varphi$, i.e. that

$$P = \frac{-1 + \sqrt{5}}{2}.$$

[15] This is a little bit informal (and we omit the details), but it should be intuitively obvious.