NDMI011: Combinatorics and Graph Theory 1

Lecture #2
Generating functions (part I)

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1 Partial fractions

We begin with an example, and then we explain the general principle. It is easy to check that

$$\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}.$$

Verifying that the equality above is correct is quite easy; but how do we compute the expression on the right, given the expression on the left? We proceed as follows. The numerator is of smaller degree than the denominator,[1] and the denominator is expressed as the product of linear terms. So, we write

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

By multiplying both sides by $x^2(x-1)$, we obtain

$$1 = (A+C)x^2 + (-A+B)x - B.$$

The left-hand-side and the right-hand-side are identical as polynomials, and so they have exactly the same coefficients. So, we get the following system of linear equations:

$$A + C = 0, \qquad -A + B = 0, \qquad -B = 1.$$

By solving the system, we obtain

$$A = -1, \qquad B = -1, \qquad C = 1,$$

[1] This is important! If the degree of the numerator is greater or equal to the degree of the denominator, then this will not work.

and we deduce that

$$\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}.$$

Now, let us try to generalize the example above. Suppose $p(x)$ and $q(x)$ are polynomials with complex coefficients[2] such that $\deg p(x) < \deg q(x)$. Next, suppose that $q(x)$ can be factored as

$$q(x) = c(x-\alpha_1)^{\beta_1} \dots (x-\alpha_t)^{\beta_t},$$

where c is a non-zero complex number, $\alpha_1, \dots, \alpha_t$ are pairwise distinct complex numbers, and $\beta_1, \dots, \beta_t$ are positive integers.[3] In this case,[4] there exist complex numbers $A_{1,1}, \dots, A_{1,\beta_1}, \dots, A_{t,1}, \dots, A_{t,\beta_t}$ such that

$$\frac{p(x)}{q(x)} = \frac{A_{1,1}}{x-\alpha_1} + \dots + \frac{A_{1,\beta_1}}{(x-\alpha_1)^{\beta_1}} + \dots + \frac{A_{t,1}}{x-\alpha_t} + \dots + \frac{A_{t,\beta_t}}{(x-\alpha_t)^{\beta_t}}.$$

We find the numbers $A_{1,1}, \dots, A_{1,\beta_1}, \dots, A_{t,1}, \dots, A_{t,\beta_t}$ by multiplying both sides by $q(x)$, then writing the resulting polynomials on both sides in the standard form,[5] and finally, setting corresponding coefficients equal to each other. This yields a system of linear equations, and we obtain the coefficients $A_{1,1}, \dots, A_{1,\beta_1}, \dots, A_{t,1}, \dots, A_{t,\beta_t}$ by solving this system.

For example, for the rational expression $\frac{x^5-7x+1}{(x-2)^3(x+1)^2(x+2)^4}$, we would get the equation

$$\begin{aligned}&\tfrac{x^5-7x+1}{7(x-2)^3(x+1)^2(x+2)^4}\\ = \;& \tfrac{A}{x-2} + \tfrac{B}{(x-2)^2} + \tfrac{C}{(x-2)^3} + \tfrac{D}{x+1} + \tfrac{E}{(x+1)^2} + \tfrac{F}{x+2} + \tfrac{G}{(x+2)^2} + \tfrac{H}{(x+2)^3} + \tfrac{I}{(x+2)^4},\end{aligned}$$

though computing $A, \dots, I$ by hand would take quite some time.

Let us now consider a computationally easier example:

$$\frac{3x^2+4}{x^3(x+1)^2}.$$

The polynomial in the numerator is of smaller degree than the polynomial in the denominator, and so there exist numbers A, B, C, D, E such that

$$\frac{3x^2+4}{x^3(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{(x+1)^2}.$$

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[2] In examples that we consider, we will work only with real numbers. However, the method works exactly the same way for complex numbers.

[3] Note that in the example from the beginning of the section, we have $c = 1$, $t = 2$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 2$, and $\beta_2 = 1$.

[4] We omit the proof, but you can try to convince yourself that this is true.

[5] That is to say, in the form $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, where $a_n, \dots, a_0$ are complex numbers.

After multiplying both sides by $x^3(x+1)^2$, we get

$$3x^2+4 = Ax^2(x+1)^2 + Bx(x+1)^2 + C(x+1)^2 + Dx^3(x+1) + Ex^3,$$

and after writing the polynomial on the right-hand-side in standard form, we get

$$3x^2+4 = (A+D)x^4 + (2A+B+D+E)x^3 + (A+2B+C)x^2 + (B+2C)x + C$$

The polynomial on the left-hand-side and the one on the right-hand-side have the same coefficients, which yields the following system of linear equations:

$$\begin{array}{rcrcrcrcrcl} A & & & & & + & D & & & = & 0 \\ 2A & + & B & & & + & D & + & E & = & 0 \\ A & + & 2B & + & C & & & & & = & 3 \\ & & B & + & 2C & & & & & = & 0 \\ & & & & C & & & & & = & 4 \end{array}$$

By solving the system, we obtain

$$A = 15, \quad B = -8, \quad C = 4, \quad D = -15, \quad E = -7.$$

So, we have that

$$\frac{3x^2+4}{x^3(x+1)^2} = \frac{15}{x} - \frac{8}{x^2} + \frac{4}{x^3} - \frac{15}{x+1} - \frac{7}{(x+1)^2}.$$

As pointed out earlier in the section, we can perform the procedure described above only on rational expressions of the form $\frac{p(x)}{q(x)}$, where $p(x)$ has strictly smaller degree than $q(x)$. If deg $p(x) \geq$ deg $q(x)$, then we first perform polynomial division, and then we perform the procedure on the remainder. For instance,

$$\begin{aligned} \tfrac{3x^4-3x^3+1}{x^2(x-1)} &\overset{(*)}{=} 3x + \tfrac{1}{x^2(x-1)} \\ &\overset{(**)}{=} 3x - \tfrac{1}{x} - \tfrac{1}{x^2} + \tfrac{1}{x-1} \end{aligned}$$

where (*) is obtained by dividing polynomials, and (**) is from the calculation performed at the beginning of the section.

2 The Taylor series: a review

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in A$, and assume that A contains (as a subset) some open neighborhood of a.[6] Assume furthermore that f is infinitely differentiable at a.[7] Then the *Taylor series* of f centered at a is the series

$$T^{f,a}(x) \quad = \quad \sum_{n=0}^{\infty} \tfrac{f^{(n)}(a)}{n!}(x-a)^n.$$

The Taylor series $T^{f,0}(x)$ (here, we have $a = 0$) is called the *Maclaurin series.*

For a real number α and a non-negative integer k, we define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}.$$

In particular, $\binom{\alpha}{0} = 1$.

Here are the Maclaurin series of some familiar functions:

(i) $T^{\exp(x),0}(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots;$

(ii) $T^{\sin x,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \dots;$

(iii) $T^{\cos x,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots;$

(iv) $T^{\ln(1+x),0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + \dots;$

(v) $T^{(1+x)^\alpha,0}(x) = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \dots + \binom{\alpha}{n}x^n + \dots$, where α is a fixed real number;

(vi) $T^{\frac{1}{1-x},0}(x) = 1 + x + x^2 + \dots + x^n + \dots.$

Let us verify (v). Fix a real number α. It is easy to verify by induction[8] that for all positive integers k, we have that

$$\tfrac{d^k}{dx^k}(1+x)^\alpha \quad = \quad \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k},$$

and consequently,

$$\frac{\frac{d^k}{dx^k}(1+x)^\alpha\Big|_{x=0}}{k!} \quad = \quad \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \quad = \quad \binom{\alpha}{k},$$

where as usual, $\frac{d^k}{dx^k}(1+x)^\alpha$ denotes the k-th derivative of the function $(1+x)^\alpha$,[9] and $\frac{d^k}{dx^k}(1+x)^\alpha\Big|_{x=0}$ is the k-th derivative of $(1+x)^\alpha$ evaluated at $x = 0$. So, (v) holds.

[6] So, there exists some $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq A$.

[7] f is *infinitely differentiable* at a if $f^{(n)}(a)$ exists for all $n \geq 0$. (In particular, f is differentiable, and therefore continuous, at a.)

[8] Check this!

[9] The zeroth derivative of a function is simply the function itself.

We remark that these series do not necessarily converge for all values of x. Furthermore, in general, it is possible that $T^{f,a}(x)$ converges, but does **not** converge to $f(x)$. Nonetheless, we do have the following:

(1) $\exp(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \ldots$ for all $x \in \mathbb{R}$;

(2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots$ for all $x \in \mathbb{R}$;

(3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{\frac{x^{2n}}{(2n)!}} + \ldots$ for all $x \in \mathbb{R}$;

(4) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \ldots$ for all $x \in (-1, 1]$;

(5) $(1 + x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \ldots$ for $x \in (-1, 1)$, where α is a fixed real number;

(6) $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \ldots$ for $x \in (-1, 1)$.

For a non-zero constant a, a positive integer t, and a sufficiently small value of x, we can substitute ax^t for x in the above equations. So, for example, by substituting $2x^3$ for x in (6), we get that

$$\frac{1}{1-2x^3} = 1 + 2x^3 + 4x^6 + \cdots + 2^n x^{3n} + \ldots$$

(as long as x is sufficiently small). In working with generating functions (see the section 3 below), we will not worry about exactly how small x needs to be to make our equations work; we simply need that they work for values of x in some (no matter how small) open neighborhood of zero. We also note that (6) follows from (5) for $\alpha = -1$, with $-x$ substituted for x; indeed,

$$\begin{aligned}
\frac{1}{1-x} &= (1 - (-x))^{-1} \\
&= \sum_{n=0}^{\infty} \binom{-1}{n}(-x)^n && \text{by (5)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)(-2)\ldots(-1-n+1)}{n!}(-x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(-1)^n x^n \\
&= \sum_{n=0}^{\infty} x^n,
\end{aligned}$$

which is precisely (6).

Finally, we remark that the identity from (5) is sometimes called the "Generalized Binomial Theorem." Note that if α is a non-negative integer, then $\binom{\alpha}{k} = 0$ for all integers $k > \alpha$, and we get that

$$(1 + x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \cdots + \binom{\alpha}{\alpha}x^\alpha,$$

which is what we also get from the usual (finite) Binomial Theorem. However, if α is negative or not an integer, then the series from (5) is indeed infinite.

3 Generating functions

3.1 A motivating example

We motivate our study of generating function with the following question: How many ways are there to pay 21 Kč, assuming we have six 1 Kč coins, five 2 Kč coins, and four 5 Kč coins?[10] Here, we are looking for the number of solutions to the equation $i_1 + i_2 + i_5 = 21$, with $i_1 \in \{0, 1, 2, 3, 4, 5, 6\}$, $i_2 \in \{0, 2, 4, 6, 8, 10\}$, and $i_5 \in \{0, 5, 10, 15, 20\}$. Indeed, i_1 is the amount paid with 1 Kč coins, i_2 is the sum paid with 2 Kč coins, and i_5 is the amount paid with 5 Kč coins. Now, we note that the number of solutions is precisely the coefficient in front of x^{21} in the following polynomial:

$$\begin{aligned} p(x) &= (1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10}) \\ &\quad \times (1 + x^5 + x^{10} + x^{15} + x^{20}) \end{aligned}$$

Indeed, we obtain x^{21} by selecting some x^{i_1} from the first term of the product, some x^{i_2} from the second, and some x^{i_5} from the third, in such a way that $i_1 + i_2 + i_5 = 21$. The number of ways of selecting i_1, i_2, i_5 is precisely the coefficient in front of x^{21} in the polynomial $p(x)$. By using computer software,[11] we see that this coefficient is 9. So, there are 9 ways to make our payment. More generally, for each non-negative integer n, let a_n be the number of ways to pay nKč using our coins; then a_n is precisely the coefficient in front of x^n in the polynomial $p(x)$, i.e.

$$p(x) = \sum_{n=0}^{\infty} a_n x^n$$

We call $p(x)$ the "generating function" of the sequence $\{a_n\}_{n=0}^{\infty}$. In this case, it is a polynomial,[12] but in general, it is a (potentially infinite) series. (A formal definition of a generating function is given in section 3.2 below).

It might seem that the use of polynomials in the example above does not simplify the problem. Indeed, if you compute by hand, it is easier to simply enumerate all the solutions. However, polynomials are more convenient if we wish to use a computer. More importantly, we can use a similar idea to solve more complicated problems.

[10] Here, we assume that all coins of the same value are the same. So, if we happened to use three 1 Kč coins, we do not care which particular three we chose.

[11] Or by hand, if you are in the mood to compute.

[12] This is because we only have 36 Kč, and so $a_n = 0$ for all integers $n \geq 37$.

3.2 Generating functions as power series

Suppose $\{a_n\}_{n=0}^{\infty}$ is some infinite sequence of real numbers.[13] The *generating function* of this sequence is the power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

For example, the generating function of the constant sequence $1, 1, 1, 1, 1, \ldots$ is

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

We recognize the above sequence as the Maclaurin series of the function $\frac{1}{1-x}$. So, the generating function of $1, 1, 1, 1, 1, \ldots$ is $\frac{1}{1-x}$.

3.3 Generating functions and recursively defined sequences

For a positive integer k, a *homogeneous linear difference equation of degree k* is an equation of the form

$$y_{n+k} = a_{k-1}y_{n+k-1} + a_{k-2}y_{n+k-2} + \cdots + a_1 y_{n+1} + a_0 y_n,$$

where $a_{k-1}, \ldots, a_0$ are fixed constants. Often, sequences are defined by specifying the values of the first k terms, and by a homogeneous linear difference equation of degree k. One famous example of such a sequence is the *Fibonacci sequence* $\{F_n\}_{n=0}^{\infty}$, defined recursively as follows:

- $F_0 = 0$, $F_1 = 1$;
- $F_{n+2} = F_n + F_{n+1}$ for all integers $n \geq 0$.

(Numbers F_n are called the *Fibonacci numbers*.) So, we defined the Fibonacci sequence using a second degree homogeneous linear difference equation.

In theory, generating functions can be used to find the closed formula of the general term of a sequence defined via any homogeneous linear difference equation. However, in practice, if our difference equation is of high degree, this may be difficult or impossible to do due to problems with factoring polynomials of high degree.[14] Here, we show how this can be done for sequences defined via second degree homogeneous linear difference equations. We begin with the Fibonacci sequence.

Example 3.1. *Find a closed formula for F_n $(n \geq 0)$, where F_n is the n-th Fibonacci number.*

[13] Actually, this also works for complex numbers.

[14] The quadratic equation allows us to easily factor second degree polynomials. There are also formulas for factoring third and fourth degree polynomials. However, there is no general formula for factoring fifth (and higher) degree polynomials.

Solution. We consider the generating function

$$f(x) = \sum_{n=0}^{\infty} F_n x^n$$

for $\{F_n\}_{n=0}^{\infty}$. We now manipulate this function as follows:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} F_n x^n \\
&= F_0 + F_1 x + x^2 \sum_{n=0}^{\infty} F_{n+2} x^n \\
&= x + x^2 \sum_{n=0}^{\infty} (F_n + F_{n+1}) x^n \\
&= x + (x^2 \sum_{n=0}^{\infty} F_n x^n) + (x^2 \sum_{n=0}^{\infty} F_{n+1} x^n) \\
&= x + (x^2 \sum_{n=0}^{\infty} F_n x^n) + (x \sum_{n=0}^{\infty} F_{n+1} x^{n+1}) \\
&= x + (x^2 \sum_{n=0}^{\infty} F_n x^n) + (x \sum_{n=0}^{\infty} F_n x^n) \qquad \text{because } F_0 = 0 \\
&= x + x^2 f(x) + x f(x)
\end{aligned}$$

So, we have obtained the equation

$$f(x) = x + x^2 f(x) + x f(x),$$

which, in turn, yields

$$f(x) = -\frac{x}{x^2 + x - 1}.$$

We now compute:

$$\begin{array}{rcll}
f(x) & = & -\frac{x}{x^2+x-1} & \\
& = & -\frac{x}{(x-\frac{-1-\sqrt{5}}{2})(x-\frac{-1+\sqrt{5}}{2})} & \text{via the quadratic equation} \\
& = & -\frac{\frac{1+\sqrt{5}}{2\sqrt{5}}}{x-\frac{-1-\sqrt{5}}{2}} - \frac{\frac{-1+\sqrt{5}}{2\sqrt{5}}}{x-\frac{-1+\sqrt{5}}{2}} & \text{via partial fractions} \\
& = & -\frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{2}}{x-\frac{-1-\sqrt{5}}{2}} + \frac{\frac{-1+\sqrt{5}}{2}}{x-\frac{-1+\sqrt{5}}{2}}\right) & \\
& = & -\frac{1}{\sqrt{5}}\left(\frac{1}{1+x\frac{2}{1+\sqrt{5}}} - \frac{1}{1+x\frac{2}{1-\sqrt{5}}}\right) & \\
& = & -\frac{1}{\sqrt{5}}\left(\frac{1}{1-x\frac{1-\sqrt{5}}{2}} - \frac{1}{1-x\frac{1+\sqrt{5}}{2}}\right) & \\
& = & \frac{1}{\sqrt{5}}\Big((-\sum_{n=0}^{\infty}(\frac{1-\sqrt{5}}{2})^n x^n) + (\sum_{n=0}^{\infty}(\frac{1+\sqrt{5}}{2})^n x^n)\Big) & \text{via Maclaurin expansion} \\
& = & \sum_{n=0}^{\infty}\frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}x^n &
\end{array}$$

Recall that $f(x) = \sum_{n=0}^{\infty} F_n x^n$. So, for all nonnegative integers n, we have that

$$F_n = \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

We can easily check that the answer is correct by induction. Indeed,

$$\frac{(1+\sqrt{5})^0-(1-\sqrt{5})^0}{2^0\sqrt{5}} = 0 = F_0$$

$$\frac{(1+\sqrt{5})^1-(1-\sqrt{5})^1}{2^1\sqrt{5}} = 1 = F_1$$

and so the formula is correct for $n = 0$ and $n = 1$. For the induction step, we fix an integer $n \geq 0$, and we assume that

$$F_n = \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}};$$

$$F_{n+1} = \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}.$$

Then

$$\begin{aligned}
F_{n+2} &= F_n + F_{n+1} \\
&= \tfrac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}} + \tfrac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} \\
&= \tfrac{1}{\sqrt{5}}\Big((\tfrac{1+\sqrt{5}}{2})^n(1+\tfrac{1+\sqrt{5}}{2}) - (\tfrac{1-\sqrt{5}}{2})^n(1+\tfrac{1-\sqrt{5}}{2})\Big) \\
&= \tfrac{1}{\sqrt{5}}\Big((\tfrac{1+\sqrt{5}}{2})^n\tfrac{3+\sqrt{5}}{2} - (\tfrac{1-\sqrt{5}}{2})^n\tfrac{3-\sqrt{5}}{2}\Big) \\
&= \tfrac{1}{\sqrt{5}}\Big((\tfrac{1+\sqrt{5}}{2})^n(\tfrac{1+\sqrt{5}}{2})^2 - (\tfrac{1-\sqrt{5}}{2})^n(\tfrac{1-\sqrt{5}}{2})^2\Big) \\
&= \tfrac{1}{\sqrt{5}}\Big((\tfrac{1+\sqrt{5}}{2})^{n+2} - (\tfrac{1-\sqrt{5}}{2})^{n+2}\Big) \\
&= \tfrac{(1+\sqrt{5})^{n+2}-(1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}},
\end{aligned}$$

and so the formula is correct for $n+2$. □

The *golden ratio* is the number

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

Our solution to Example 3.1 implies that the n-th Fibonacci number ($n \geq 0$) satisfies[15]

$$F_n = \tfrac{\varphi^n-(1-\varphi)^n}{\sqrt{5}} = \tfrac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}.$$

Example 3.2. *Let* $\{a_0\}_{n=0}^{\infty}$ *be a sequence defined recursively as follows:*

- $a_0 = 0$ *and* $a_1 = 1$*;*
- $a_{n+2} = -a_n + 2a_{n+1}$ *for all integers* $n \geq 0$*.*

Find a closed formula for a_n *(*$n \geq 0$*).*

Solution. We consider the generating function

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

[15]Check this!

for the sequence $\{a_n\}_{n=0}^{\infty}$, and we compute:

$$\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + x^2 \sum_{n=0}^{\infty} a_{n+2} x^n \\
&= x + x^2 \sum_{n=0}^{\infty} (-a_n + 2a_{n+1}) x^n \\
&= x - \left(x^2 \sum_{n=0}^{\infty} a_n\right) + \left(2x \sum_{n=0}^{\infty} a_{n+1} x^{n+1}\right) \\
&= x - \left(x^2 \sum_{n=0}^{\infty} a_n\right) + \left(2x \sum_{n=0}^{\infty} a_n x^n\right) && \text{because } a_0 = 0 \\
&= x - x^2 a(x) + 2xa(x)
\end{aligned}$$

Thus, we obtained the equation

$$a(x) = x - x^2 a(x) + 2xa(x),$$

which yields

$$a(x) = \frac{x}{(x-1)^2}$$

We now compute:

$$\begin{aligned}
a(x) &= \tfrac{x}{(x-1)^2} \\
&= -\tfrac{1}{1-x} + \tfrac{1}{(1-x)^2} && \text{via partial fractions} \\
&= -\left(\sum_{n=0}^{\infty} x^n\right) + \left(\sum_{n=0}^{\infty} \tbinom{-2}{n} (-x)^n\right) && \text{via Maclaurin expansion} \\
&= -\left(\sum_{n=0}^{\infty} x^n\right) + \left(\sum_{n=0}^{\infty} \tfrac{(-2)(-3)\ldots(-2-n+1)}{n!} (-x)^n\right) \\
&= -\left(\sum_{n=0}^{\infty} x^n\right) + \left(\sum_{n=0}^{\infty} \tfrac{(-1)^n (n+1)!}{n!} (-1)^n x^n\right) \\
&= -\left(\sum_{n=0}^{\infty} x^n\right) + \left(\sum_{n=0}^{\infty} (n+1) x^n\right) \\
&= \sum_{n=0}^{\infty} n x^n
\end{aligned}$$

Since $a(x) = \sum_{n=0}^{\infty} a_n$, we deduce that $a_n = n$ for all integers $n \geq 0$.[16]

We can easily check that our formula is correct by induction. Indeed, $a_0 = 0$ and $a_1 = 1$ by construction, and so the formula is correct for $n = 0$ and $n = 1$. For the induction step, we fix an integer $n \geq 0$, we assume inductively that $a_n = n$ and $a_{n+1} = n + 1$, and we observe that

$$\begin{aligned} a_{n+2} &= -a_n + 2a_{n+1} \\ &= -n + 2(n+1) \\ &= n + 2, \end{aligned}$$

and so the formula is correct for $n + 2$. This completes the induction. $\square$

Sometimes, generating functions can be used to find a closed formula for the general term of a recursively defined sequence, even if the recurrence is not given by a homogeneous linear difference equation. We now look at one such example.

Example 3.3. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence defined recursively as follows:*

- $a_0 = 1$*;*
- $a_{n+1} = 7a_n + 6^{n+1}$ *for all integers* $n \geq 0$*.*

Find a closed formula for a_n.

[16] Alternatively, we could have proceeded as follows:

$$\begin{aligned} a(x) &= \tfrac{x}{(x-1)^2} \\ &= x\tfrac{1}{(1-x)^2} \\ &= x \sum_{n=0}^{\infty} \tbinom{-2}{n}(-x)^n && \text{via Maclaurin expansion} \\ &= x \sum_{n=0}^{\infty} \tfrac{(-2)(-3)\dots(-2-n+1)}{n!}(-x)^n \\ &= x \sum_{n=0}^{\infty} \tfrac{(-1)^n(n+1)!}{n!}(-1)^n x^n \\ &= x \sum_{n=0}^{\infty} (n+1)x^n \\ &= \sum_{n=0}^{\infty} (n+1)x^{n+1} \\ &= \sum_{n=0}^{\infty} nx^n, \end{aligned}$$

and so $a_n = n$ for all integers $n \geq 0$.

Solution. We consider the generating function

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

for the sequence $\{a_n\}_{n=0}^{\infty}$. We manipulate this function as follows:

$$\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=0}^{\infty} a_{n+1} x^{n+1} \\
&= 1 + \sum_{n=0}^{\infty} (7a_n + 6^{n+1}) x^{n+1} \\
&= 1 + 7x\Big(\sum_{n=0}^{\infty} a_n x^n\Big) + \Big(\sum_{n=1}^{\infty} 6^n x^n\Big) \\
&= 7x\Big(\sum_{n=0}^{\infty} a_n x^n\Big) + \Big(\sum_{n=0}^{\infty} 6^n x^n\Big) \\
&= 7xa(x) + \tfrac{1}{1-6x}.
\end{aligned}$$

So, we have obtained the equation

$$a(x) = 7xa(x) + \frac{1}{1-6x},$$

which implies that

$$a(x) = \frac{1}{(7x-1)(6x-1)}.$$

We now compute

$$\begin{aligned}
a(x) &= \tfrac{1}{(7x-1)(6x-1)} \\
&= \tfrac{7}{1-7x} - \tfrac{6}{1-6x} && \text{via partial fractions} \\
&= (7\sum_{n=0}^{\infty} 7^n x^n) - (6\sum_{n=0}^{\infty} 6^n x^n) \\
&= \sum_{n=0}^{\infty} (7^{n+1} - 6^{n+1}) x^n.
\end{aligned}$$

Recall that $a(x) = \sum_{n=0}^{\infty} a_n x^n$. So, we get that

$$a_n = 7^{n+1} - 6^{n+1}$$

for all integers $n \geq 0$.

We can check that this formula is correct by induction. Clearly,

$$7^{1+1} - 6^{1+1} \quad = \quad 1 \quad = \quad a_0,$$

and so the formula is correct for $n = 0$. Now, fix a non-negative integer n, and assume that $a_n = 7^{n+1} - 6^{n+1}$. Then

$$\begin{aligned} a_{n+1} &= 7a_n + 6^{n+1} \\ &= 7(7^{n+1} - 6^{n+1}) + 6^{n+1} \\ &= 7^{n+2} - 7 \cdot 6^{n+1} + 6^{n+1} \\ &= 7^{n+2} - 6^{n+2}. \end{aligned}$$

This completes the induction. $\square$