NDMI011: Combinatorics and Graph Theory 1

Lecture #1

Estimates of factorials and binomial coefficients

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September 29, 2020

Definition

For a positive integer n, we define $n!$ (read "n factorial") to be

$$n! \quad := \quad n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1.$$

Furthermore, as a convention, we set $0! = 1$.

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- $n!$ is the number of ways that n distinct objects can be arranged in a sequence.
    - there are n choices for the first term of the sequence, $n-1$ choices for the second, $n-2$ for the third, etc.
- For instance, there are $3! = 6$ ways to arrange the elements of $\{a, b, c\}$ in a sequence, namely:

(1) a, b, c
(2) a, c, b
(3) b, a, c
(4) b, c, a
(5) c, a, b
(6) c, b, a

- For small values of n, computing $n!$ is quite straightforward:
  - $0! = 1$
  - $1! = 1$
  - $2! = 2 \cdot 1 = 2$
  - $3! = 3 \cdot 2 \cdot 1 = 6$
  - $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
  - $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
  - $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
  - $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$
  - $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$
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- However, $n!$ is a very fast growing function, and so computing it for even moderately large n is impractical.
- How about some estimates (upper and lower bounds)?

- Obviously, $n! \leq n^n$.

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- Our goal is to obtain two better estimates for $n!$, as follows:

  (i) $n^{n/2} \leq n! \leq (\frac{n+1}{2})^n$ for all non-negative integers n;

  (ii) $e(\frac{n}{e})^n \leq n! \leq en(\frac{n}{e})^n$ for all positive integers n.

- We first prove (i).

Theorem 1.1

For all non-negative integers n, the following holds:

$$n^{n/2} \quad \leq \quad n! \quad \leq \quad \left(\frac{n+1}{2}\right)^n$$

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- We'll need the inequality of arithmetic and geometric means.

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Inequality of arithmetic and geometric means

All non-negative real numbers x and y satisfy

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Proof: Lecture Notes.

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Proof of the upper bound.

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Proof of the upper bound. The statement is obviously true for $n = 0$ and $n = 1$. For an integer $n \geq 2$:

$$\begin{aligned}
n! \quad &= \quad \sqrt{\Big(n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1\Big)\Big(1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n\Big)} \\
&= \quad \Big(\sqrt{n \cdot 1}\Big)\Big(\sqrt{(n-1) \cdot 2}\Big) \dots \Big(\sqrt{2 \cdot (n-1)}\Big)\Big(\sqrt{1 \cdot n}\Big) \\
&\overset{\text{GM}\leq\text{AM}}{\leq} \quad \tfrac{n+1}{2} \cdot \tfrac{(n-1)+2}{2} \cdot \dots \cdot \tfrac{2+(n-1)}{2} \cdot \tfrac{1+n}{2} \\
&= \quad \left(\tfrac{n+1}{2}\right)^n.
\end{aligned}$$

- We now prove (ii).

Theorem 1.3

For all positive integers n, the following holds:

$$e\left(\frac{n}{e}\right)^n \quad \leq \quad n! \quad \leq \quad en\left(\frac{n}{e}\right)^n.$$

- We now prove (ii).

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- In fact, we'll only prove the upper bound (see the Lecture Notes for the lower bound).
- We will use the following inequality (which can be proven using calculus).

Proposition 1.2

For all real numbers x, we have $1 + x \leq e^x$.

Proof: Lecture Notes.

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Proof of the upper bound.

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Proof of the upper bound. By induction on n. The statement is obviously true for $n = 1$.

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Proof of the upper bound. By induction on n. The statement is obviously true for $n = 1$. Now fix a positive integer n, and assume $n! \leq en(\frac{n}{e})^n$. WTS $(n+1)! \leq e(n+1)(\frac{n+1}{e})^{n+1}$.

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$$\begin{aligned}
(n+1)! &= (n+1) \cdot n! \\
&\leq (n+1) \cdot en(\tfrac{n}{e})^n && \text{by ind. hyp.} \\
&= \Big(e(n+1)(\tfrac{n+1}{e})^{n+1}\Big) \cdot (\tfrac{n}{n+1})^{n+1} e.
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\end{aligned}$$

It remains to show that $(\frac{n}{n+1})^{n+1} e \leq 1$.

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Proof of the upper bound (continued). WTS $(\frac{n}{n+1})^{n+1}e \leq 1$.

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Proof of the upper bound (continued). WTS $(\frac{n}{n+1})^{n+1} e \leq 1$.

$$\begin{aligned}
\left(\frac{n}{n+1}\right)^{n+1} e &= \left(1 - \frac{1}{n+1}\right)^{n+1} e \\
&\leq \left(e^{-\frac{1}{n+1}}\right)^{n+1} e \qquad \text{by Proposition 1.2 } (1 + x \leq e^x \ \forall x \in \mathbb{R}) \text{ for } x = -\frac{1}{n+1} \\
&= 1.
\end{aligned}$$

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Stirling's formula

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}\,(\frac{n}{e})^n}{n!} = 1.$$

Proof omitted.

Definition

For integers n and k such that $n \geq k \geq 0$, we define $\binom{n}{k}$, read "n choose k," as follows:

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k\cdot(k-1)\cdot\dots\cdot 1} = \prod_{i=0}^{k-1} \frac{n-i}{k-i}.$$

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- Remark: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\binom{n}{k} = \binom{n}{n-k}$.

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- Remark: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\binom{n}{k} = \binom{n}{n-k}$.
- $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

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- Remark: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\binom{n}{k} = \binom{n}{n-k}$.
- $\binom{n}{k}$ is the number of k-element subsets of an n-element set.
- For example, the number of 3-element subsets of the 5-element set $\{a, b, c, d, e\}$ is $\binom{5}{3} = 10$:

1. $\{a, b, c\}$
2. $\{a, b, d\}$
3. $\{a, b, e\}$
4. $\{a, c, d\}$
5. $\{a, c, e\}$
6. $\{a, d, e\}$
7. $\{b, c, d\}$
8. $\{b, c, e\}$
9. $\{b, d, e\}$
10. $\{c, d, e\}$

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Binomial theorem

For all integers $n \geq 0$, and all real numbers x and y, the following holds:

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \cdots + \binom{n}{n-1} x^{n-1} y + \binom{n}{n} x^n.\end{aligned}$$

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- As in the case of factorials, binomial coefficients are easy to compute for small values of n and k. However, even for moderately large n, k, computing $\binom{n}{k}$ becomes impractical.
- So, as in the case of factorials, we would like to obtain some useful estimates (convenient upper and lower bounds) for binomial coefficients.

Theorem 2.1

For all integers n and k such that $n \geq k \geq 1$, the following holds:

$$\left(\frac{n}{k}\right)^k \quad \leq \quad \binom{n}{k} \quad \leq \quad \left(\frac{en}{k}\right)^k.$$

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- Theorem 2.1 follows from the two propositions below.

Proposition 2.2

For all integers n and k such that $n \geq k \geq 1$, we have that

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Proposition 2.3

For all integers n and k such that $n \geq k \geq 1$, we have that:

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Proof. Fix integers n, k such that $n \geq k \geq 1$. We observe that for all $i \in \{0, \ldots, k-1\}$, we have that $\frac{n-i}{k-i} \geq \frac{n}{k}$,

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$$\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \geq \prod_{i=0}^{k-1} \frac{n}{k} = \left(\frac{n}{k}\right)^k.$$

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Proof. ***Claim.*** *For all real numbers x such that $0 < x \leq 1$:*

$$\sum_{i=0}^{k} \binom{n}{i} \quad \leq \quad \frac{(1+x)^n}{x^k}.$$

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Proof of the Claim. By the Binomial theorem:

$$(1+x)^n \quad = \quad \sum_{i=0}^{n} \binom{n}{i} x^i \quad \geq \quad \sum_{i=0}^{k} \binom{n}{i} x^i.$$

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Dividing by x^k, we then obtain

$$\frac{(1+x)^n}{x^k} \geq \sum_{i=0}^{k} \binom{n}{i} \frac{1}{x^{k-i}} \overset{0<x\leq 1}{\geq} \sum_{i=0}^{k} \binom{n}{i}$$

This proves the Claim.

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We now compute apply the Claim to $x := \frac{k}{n}$, and we obtain

$$\begin{aligned}
\sum_{i=0}^{k} \binom{n}{i} \quad &\leq \quad (1 + \tfrac{k}{n})^n (\tfrac{n}{k})^k && \text{by the Claim for } x = \tfrac{k}{n} \\
&\leq \quad (e^{k/n})^n (\tfrac{n}{k})^k && \text{by Proposition 1.2 for } x = \tfrac{k}{n} \\
& && (1 + x \leq e^x \ \forall x \in \mathbb{R}) \\
&= \quad (\tfrac{en}{k})^k.
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- So, we have proven Theorem 2.1.

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- Our next goal is to find a good estimate for the largest among the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.
- Which one is the largest?

- For all integers n and k such that $n \geq k \geq 1$, we have that

$$\binom{n}{k} = \binom{n}{k-1} \cdot \frac{n-k+1}{k}.$$

- For all integers n and k such that $n \geq k \geq 1$, we have that

$$\binom{n}{k} \quad = \quad \binom{n}{k-1} \cdot \frac{n-k+1}{k}.$$

- So, for even n:

$$\binom{n}{0} \quad < \quad \binom{n}{1} \quad < \quad \cdots \quad < \quad \binom{n}{n/2} \quad > \quad \cdots \quad > \quad \binom{n}{n-1} \quad > \quad \binom{n}{n},$$

- For all integers n and k such that $n \geq k \geq 1$, we have that

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- whereas for odd n:

$$\binom{n}{0} \quad < \quad \ldots \quad < \quad \binom{n}{\lfloor n/2 \rfloor} \quad = \quad \binom{n}{\lceil n/2 \rceil} \quad > \quad \ldots \quad > \binom{n}{n}.$$

- For all integers n and k such that $n \geq k \geq 1$, we have that

$$\binom{n}{k} = \binom{n}{k-1} \cdot \frac{n-k+1}{k}.$$

- So, for even n:

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- Let's find good bounds for $\binom{2m}{m}$.

Theorem 2.4

For all integers $m \geq 1$, we have that

$$\frac{2^{2m}}{2\sqrt{m}} \leq \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{2m}}$$

Proof.

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Proof. Fix an integer $m \geq 1$, and let

$$P = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2m)}.$$

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Then

$$\begin{aligned} P \quad &= \quad \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)} \\ &= \quad \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)} \cdot \frac{2 \cdot 4 \cdot \cdots \cdot (2m)}{2 \cdot 4 \cdot \cdots \cdot (2m)} \\ &= \quad \frac{(2m)!}{2^{2m}(m!)^2} \\ &= \quad \frac{1}{2^{2m}} \binom{2m}{m}. \end{aligned}$$

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We first prove the upper bound for P, as follows:

$$\begin{aligned}
1 &\geq (1-\tfrac{1}{2^2})(1-\tfrac{1}{4^2})\dots(1-\tfrac{1}{(2m)^2}) \\
&= \frac{2^2-1}{2^2}\cdot\frac{4^2-1}{4^2}\cdot\dots\cdot\frac{(2m)^2-1}{(2m)^2} \\
&= \frac{1\cdot 3}{2^2}\cdot\frac{3\cdot 5}{4^2}\cdot\dots\cdot\frac{(2m-1)(2m+1)}{(2m)^2} \\
&= (2m+1)P^2,
\end{aligned}$$

which implies $P \leq \frac{1}{\sqrt{2m+1}} \leq \frac{1}{\sqrt{2m}}$.

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which implies $P \leq \frac{1}{\sqrt{2m+1}} \leq \frac{1}{\sqrt{2m}}$. Lower bound: Lecture Notes.

- Finally, we note that using Stirling's formula (which we stated without proof), we can obtain an even better approximation of $\binom{2m}{m}$, as follows:

$$\lim_{m \to \infty} \left(\left(\frac{2^{2m}}{\sqrt{\pi m}} \right) / \binom{2m}{m} \right) = 1.$$

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- So, for very large values of m, the function $g(m) = \frac{2^{2m}}{\sqrt{\pi m}}$ is a good approximation of $\binom{2m}{m}$.