NDMI011: Combinatorics and Graph Theory 1

Lecture #1
Estimates of factorials and binomial coefficients

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1 Estimating factorials

For a positive integer n, we define $n!$ (read "n factorial") to be

$$n! \quad := \quad n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1.$$

Furthermore, as a convention, we set $0! = 1$.

$n!$ is the number of ways that n distinct objects can be arranged in a sequence: there are n choices for the first term of the sequence, $n-1$ choices for the second, $n-2$ for the third, etc. For instance, there are $3! = 6$ ways to arrange the elements of the set $\{a, b, c\}$ in a sequence, namely:

(1) a, b, c	(3) b, a, c	(5) c, a, b
(2) a, c, b	(4) b, c, a	(6) c, b, a

For small values of n, computing $n!$ is quite straightforward:

- $0! = 1$
- $1! = 1$
- $2! = 2 \cdot 1 = 2$
- $3! = 3 \cdot 2 \cdot 1 = 6$
- $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
- $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
- $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
- $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

- $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$
- $9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362880$

etc. However, as we see from the list above, $n!$ is a very fast increasing function, and computing it for even moderately large n is impractical. Nevertheless, in applications, it is often useful to know roughly how big $n!$ is, that is, how it compares to various other functions of n. Obviously,[1]

$$n! \quad \leq \quad n^n$$

for all non-negative integers n. In this lecture, we will obtain two better estimates for $n!$, as follows:

(i) $n^{n/2} \leq n! \leq (\frac{n+1}{2})^n$ for all non-negative integers n;

(ii) $e(\frac{n}{e})^n \leq n! \leq en(\frac{n}{e})^n$ for all positive integers n.

For non-negative real numbers x and y, the *arithmetic mean* of x and y is $\frac{x+y}{2}$, and the *geometric mean* of x and y is $\sqrt{xy}$. To prove (i), we will use the inequality of arithmetic and geometric means (below).

Inequality of arithmetic and geometric means. *All non-negative real numbers x and y satisfy*

$$\sqrt{xy} \quad \leq \quad \tfrac{x+y}{2}.$$

Proof. For non-negative real numbers x and y, we have the following sequence of equivalences:

$$\begin{array}{rrcl} & (\sqrt{x} - \sqrt{y})^2 & \geq & 0 \\ \iff & x - 2\sqrt{xy} + y & \geq & 0 \\ \iff & x + y & \geq & 2\sqrt{xy} \\ \iff & \frac{x+y}{2} & \geq & \sqrt{xy}. \end{array}$$

Since the first inequality above is obviously true, so is the last one. □

We are now ready to prove (i).

Theorem 1.1. *For all non-negative integers n, the following holds:*

$$n^{n/2} \quad \leq \quad n! \quad \leq \quad (\tfrac{n+1}{2})^n$$

[1]Recall that for all real numbers r, we have that $r^0 = 1$. In particular, $0^0 = 1$.

Proof. For $n = 0$ and $n = 1$, the statement is obviously true. So, fix an integer $n \geq 2$.

We first prove the upper bound, as follows:

$$\begin{aligned} n! &= \sqrt{\Big(n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1\Big)\Big(1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n\Big)} \\ &= \sqrt{\Big(n \cdot 1\Big)\Big((n-1) \cdot 2\Big) \dots \Big(2 \cdot (n-1)\Big)\Big(1 \cdot n\Big)} \\ &= \Big(\sqrt{n \cdot 1}\Big)\Big(\sqrt{(n-1) \cdot 2}\Big) \dots \Big(\sqrt{2 \cdot (n-1)}\Big)\Big(\sqrt{1 \cdot n}\Big) \\ &\overset{(*)}{\leq} \tfrac{n+1}{2} \cdot \tfrac{(n-1)+2}{2} \cdot \dots \cdot \tfrac{2+(n-1)}{2} \cdot \tfrac{1+n}{2} \\ &= (\tfrac{n+1}{2})^n, \end{aligned}$$

where (*) follows from the inequality of arithmetic and geometric means.

It remains to prove the lower bound. First, we claim that for all $i \in \{1, \dots, n\}$, we have that

$$i(n+1-i) \quad \geq \quad n.$$

Indeed, if $i = 1$ or $i = n$, then $i(n+1-i) = n$. On the other hand, for $i \in \{2, \dots, n-1\}$, we have that $\min\{i, n+1-i\} \geq 2$ and $\max\{i, n+1-i\} \geq \frac{i+(n+1-i)}{2} \geq \frac{n}{2}$, and consequently,

$$i(n+1-i) \quad = \quad \min\{i, n+1-i\} \cdot \max\{i, n+1-i\} \quad \geq \quad 2 \cdot \tfrac{n}{2} \quad = \quad n,$$

as we had claimed. We now compute:

$$\begin{aligned} n! &= \sqrt{\Big(1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n\Big)\Big(n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1\Big)} \\ &= \sqrt{\Big(1 \cdot n\Big)\Big(2 \cdot (n-1)\Big) \cdot \dots \cdot \Big(2 \cdot (n-1)\Big)\Big(1 \cdot n\Big)} \\ &= \sqrt{\prod_{i=1}^{n} \Big(\underbrace{i \cdot (n+1-i)}_{\geq n}\Big)} \\ &\geq \sqrt{n^n} \\ &= n^{n/2}, \end{aligned}$$

which is what we needed. □

It remains to prove (ii). We begin with the following proposition.

Proposition 1.2. *For all real numbers x, we have*

$$1 + x \leq e^x.$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = e^x - x - 1$. Then $f'(x) = e^x - 1$, and we have the following table:

x	$-\infty$ … $(-\infty, 0)$	0	$(0, +\infty)$ … $+\infty$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	min	$\nearrow$

So, $f(x)$ reaches a global minimum at $x = 0$, and we have that $f(0) = 0$. So, $f(x) \geq 0$ for all $x \in \mathbb{R}$, and the result follows. $\square$

We will also need the well-known fact that

$$(1 + \frac{1}{n})^n \leq e$$

for all positive integers n.[2]

We are now ready to prove (ii).

Theorem 1.3. *For all positive integers n, the following holds:*

$$e(\tfrac{n}{e})^n \quad \leq \quad n! \quad \leq \quad en(\tfrac{n}{e})^n.$$

Proof. We proceed by induction on n. The claim is clearly true for $n = 1$. Now, fix a positive integer n, and assume inductively that $e(\frac{n}{e})^n \leq n! \leq en(\frac{n}{e})^n$. We must show that $e(\frac{n+1}{e})^{n+1} \leq (n+1)! \leq e(n+1)(\frac{n+1}{e})^{n+1}$.

We first obtain the needed upper bound, i.e. we prove that $(n+1)! \leq e(n+1)(\frac{n+1}{e})^{n+1}$. We first compute:

$$\begin{aligned} (n+1)! &= (n+1) \cdot n! \\ &\leq (n+1) \cdot en(\tfrac{n}{e})^n && \text{by the induction hypothesis} \\ &= \Big(e(n+1)(\tfrac{n+1}{e})^{n+1}\Big) \cdot (\tfrac{n}{n+1})^{n+1} e. \end{aligned}$$

[2] As you saw in Analysis, the sequence $\{(1 + \frac{1}{n})^n\}_{n=1}^{\infty}$ is strictly increasing and bounded above, and so by the Monotone Sequence Theorem, it converges. The constant e is defined as the limit of this sequence, i.e. $e := \lim_{n\to\infty}(1 + \frac{1}{n})^n$, and the inequality follows.

It now remains to show that $(\frac{n}{n+1})^{n+1}e \leq 1$, for then we will obtain precisely the inequality that we need. We obtain this as follows:

$$\begin{aligned}
(\tfrac{n}{n+1})^{n+1}e &= (1 - \tfrac{1}{n+1})^{n+1}e \\
&\leq (e^{-\frac{1}{n+1}})^{n+1}e && \text{by Proposition 1.2, for } x = -\tfrac{1}{n+1} \\
&= 1.
\end{aligned}$$

It remains to establish the lower bound, i.e. to prove that $e(\frac{n+1}{e})^{n+1} \leq (n+1)!$. For this, we compute:

$$\begin{aligned}
e(\tfrac{n+1}{e})^{n+1} &= (n+1)(\tfrac{n}{e})^n \cdot (1 + \tfrac{1}{n})^n \\
&\leq (n+1)(\tfrac{n}{e})^n \cdot e && \text{because } (1 + \tfrac{1}{n})^n \leq e \\
&\leq (n+1) \cdot n! && \text{by the induction hypothesis} \\
&= (n+1)!
\end{aligned}$$

which is what we needed. □

We complete this section by giving the following formula (without proof).

Stirling's formula. $\lim_{n\to\infty} \frac{\sqrt{2\pi n}\,(\frac{n}{e})^n}{n!} = 1.$

So, for very large values of n, the function $f(n) = \sqrt{2\pi n}\,(\frac{n}{e})^n$ is a good approximation of $n!$.

2 Estimating binomial coefficients

For integers n and k such that $n \geq k \geq 0$, we define $\binom{n}{k}$, read "n choose k," as follows:

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k\cdot(k-1)\cdot\cdots\cdot 1} = \prod_{i=0}^{k-1} \frac{n-i}{k-i}.$$

Note that this implies that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and consequently,

$$\binom{n}{k} = \binom{n}{n-k}.$$

$\binom{n}{k}$ is the number of k-element subsets of an n-element set.[3] For example, the number of 3-element subsets of the 5-element set $\{a, b, c, d, e\}$ is $\binom{5}{3} = 10$; those subsets are:

(1) $\{a, b, c\}$	(3) $\{a, b, e\}$	(5) $\{a, c, e\}$	(7) $\{b, c, d\}$	(9) $\{b, d, e\}$
(2) $\{a, b, d\}$	(4) $\{a, c, d\}$	(6) $\{a, d, e\}$	(8) $\{b, c, e\}$	(10) $\{c, d, e\}$

We note that for all non-negative integers n, we have that $\binom{n}{0} = 1$. In particular, $\binom{0}{0} = 1$.

Numbers $\binom{n}{k}$ are called *binomial coefficients*. You are already familiar with the binomial theorem (below).

Binomial theorem. *For all integers $n \geq 0$, and all real numbers x and y, the following holds:*

$$\begin{aligned}(x + y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \cdots + \binom{n}{n-1} x^{n-1} y + \binom{n}{n} x^n.\end{aligned}$$

As in the case of factorials, binomial coefficients are easy to compute for small values of n and k. However, even for moderately large n, k, computing $\binom{n}{k}$ becomes impractical. So, as in the case of factorials, we would like to obtain some useful estimates (convenient upper and lower bounds) for binomial coefficients.

2.1 Estimating the binomial coefficient $\binom{n}{k}$

Our goal is to prove the following theorem.

Theorem 2.1. *For all integers n and k such that $n \geq k \geq 1$, the following holds:*

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Theorem 2.1 readily follows from Propositions 2.2 and 2.3 (below). Proposition 2.2 establishes the lower bound from Theorem 2.1, and Proposition 2.3 establishes the upper bound.[4]

Proposition 2.2. *For all integers n and k such that $n \geq k \geq 1$, we have that*

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k}$$

[3] Indeed, there are $n(n-1)\ldots(n-k+1)$ sequences of k different elements of an n-element set: there are n ways to select the first element, $n-1$ ways to select the second element, …, and $n-k+1$ ways to select the k-th element. Since every k-element set can be ordered in $k!$ ways, there are exactly $\frac{n(n-1)\ldots(n-k+1)}{k!} = \binom{n}{k}$ many k-element subsets of an n-element set.

[4] In fact, the inequality from Proposition 2.3 is stronger than the upper bound from Theorem 2.1.

Proof. Fix integers n, k such that $n \geq k \geq 1$. We observe that for all $i \in \{0, \dots, k-1\}$, we have that $\frac{n-i}{k-i} \geq \frac{n}{k}$,[5] and so

$$\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \geq \prod_{i=0}^{k-1} \frac{n}{k} = (\tfrac{n}{k})^k,$$

which is what we needed. □

Proposition 2.3. *For all integers n and k such that $n \geq k \geq 1$, we have that:*

$$\sum_{i=0}^{k} \binom{n}{i} \leq (\tfrac{en}{k})^k.$$

Proof. Fix integers n and k such that $n \geq k \geq 1$.

> ***Claim.*** *For all real numbers x such that $0 < x \leq 1$, we have that*
>
> $$\sum_{i=0}^{k} \binom{n}{i} \leq \frac{(1+x)^n}{x^k}.$$

Proof of the Claim. Fix a real number x such that $0 < x \leq 1$. By the Binomial theorem, we have that

$$\begin{aligned} (1+x)^n &= \sum_{i=0}^{n} \binom{n}{i} x^i \\ &\geq \sum_{i=0}^{k} \binom{n}{i} x^i && \text{since } n \geq k \text{ and } x > 0 \end{aligned}$$

Dividing by x^k, we then obtain

$$\begin{aligned} \frac{(1+x)^n}{x^k} &\geq \sum_{i=0}^{k} \binom{n}{i} \frac{1}{x^{k-i}} \\ &\geq \sum_{i=0}^{k} \binom{n}{i} && \text{because } 0 < x \leq 1 \end{aligned}$$

This proves the Claim. ■

We now compute apply the Claim to $x := \frac{k}{n}$, and we obtain

$$\begin{aligned} \sum_{i=0}^{k} \binom{n}{i} &\leq (1 + \tfrac{k}{n})^n (\tfrac{n}{k})^k && \text{by the Claim for } x = \tfrac{k}{n} \\ &\leq (e^{k/n})^n (\tfrac{n}{k})^k && \text{by Proposition 1.2 for } x = \tfrac{k}{n} \\ &= (\tfrac{en}{k})^k, \end{aligned}$$

which is what we needed. □

[5] Indeed, this is equivalent to $(n-i)k \geq n(k-i)$, which is in turn equivalent to $ni \geq ki$, which is true since $n \geq k$ and $i \geq 0$.

2.2 Estimating the binomial coefficient $\binom{2n}{n}$

Note that for all integers n and k such that $n \geq k \geq 1$, we have that

$$\binom{n}{k} = \binom{n}{k-1} \cdot \frac{n-k+1}{k}.$$

This implies that[6] for even n, we have that

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{n/2} > \cdots > \binom{n}{n-1} > \binom{n}{n},$$

whereas for odd n, we have that

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n}.$$

In particular, $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$ is maximum among the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. For this reason, it is of particular interest to find good estimates for the behavior of binomial coefficients of the form $\binom{n}{\lfloor n/2 \rfloor}$.

Theorem 2.4. *For all integers $m \geq 1$, we have that*

$$\frac{2^{2m}}{2\sqrt{m}} \leq \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{2m}}$$

Proof. Fix an integer $m \geq 1$, and let

$$P = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)}.$$

Then

$$\begin{aligned} P &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)} \\ &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2m)} \cdot \frac{2 \cdot 4 \cdot \cdots \cdot (2m)}{2 \cdot 4 \cdot \cdots \cdot (2m)} \\ &= \frac{(2m)!}{2^{2m}(m!)^2} \\ &= \frac{1}{2^{2m}} \binom{2m}{m}. \end{aligned}$$

It now suffices to show that

$$\frac{1}{2\sqrt{m}} \leq P \leq \frac{1}{\sqrt{2m}},$$

for the result then follows immediately.

We first establish the upper bound for P. For this, we observe that

$$\begin{aligned} 1 &\geq (1 - \tfrac{1}{2^2})(1 - \tfrac{1}{4^2}) \ldots (1 - \tfrac{1}{(2m)^2}) \\ &= \frac{2^2-1}{2^2} \cdot \frac{4^2-1}{4^2} \cdot \cdots \cdot \frac{(2m)^2-1}{(2m)^2} \\ &= \frac{1 \cdot 3}{2^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \cdots \cdot \frac{(2m-1)(2m+1)}{(2m)^2} \\ &= (2m+1)P^2, \end{aligned}$$

[6] Check this!

and consequently, $P^2 \le \frac{1}{2m+1}$, which in turn implies that

$$P \quad \le \quad \frac{1}{\sqrt{2m+1}} \quad \le \quad \frac{1}{\sqrt{2m}},$$

which is what we needed.

It remains to establish our lower bound for P. The proof is similar as for the upper bound. We observe the following:

$$\begin{aligned} 1 \quad &\ge \quad (1-\tfrac{1}{3^2})(1-\tfrac{1}{5^2})\dots(1-\tfrac{1}{(2m-1)^2}) \\ &= \quad \tfrac{3^2-1}{3^2}\cdot\tfrac{5^2-1}{5^2}\cdot\dots\cdot\tfrac{(2m-1)^2-1}{(2m-1)^2} \\ &= \quad \tfrac{2\cdot4}{3^2}\cdot\tfrac{4\cdot6}{5^2}\cdot\dots\cdot\tfrac{(2m-2)(2m)}{(2m-1)^2} \\ &= \quad \tfrac{1}{2(2m)P^2}, \end{aligned}$$

which implies that

$$P \quad \ge \quad \frac{1}{2\sqrt{m}},$$

which is what we needed. This completes the argument. □

Finally, we note that using Stirling's formula (which we stated without proof), we can obtain an even better approximation of $\binom{2m}{m}$, as follows:

$$\lim_{m\to\infty}\left(\left(\frac{2^{2m}}{\sqrt{\pi m}}\right)/\binom{2m}{m}\right) \quad = \quad 1.$$

So, for very large values of m, the function $g(m) = \frac{2^{2m}}{\sqrt{\pi m}}$ is a good approximation of $\binom{2m}{m}$.