NDMI011: Combinatorics and Graph Theory 1
HW#5

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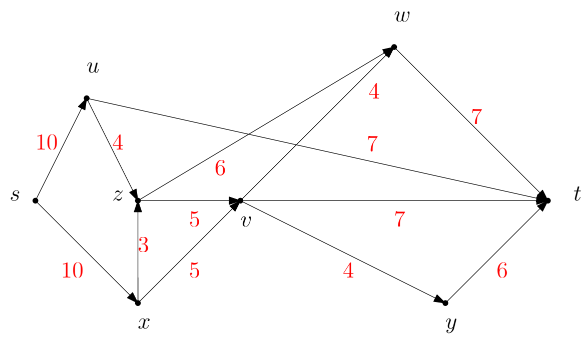
due Monday, November 9, 2020 before midnight (Prague time)

Remark: Please e-mail me (ipenev@iuuk.mff.cuni.cz) your HW as a **PDF attachment** (no other format will be accepted).

Problem 1 (25 points)**.** *Find a maximum flow and a minimum capacity cut in the network below. (As usual, s is the source, t is the sink, and capacities are in red.)*



A *multi-source, multi-sink network* is a four-tuple (G, S, T, c) where G is an oriented graph, S and T are non-empty and disjoint subsets of $V(G)$,[1] and $c : E(G) \to [0, +\infty)$ is a function called the *capacity function.*[2] A *feasible flow* (or simply *flow*) in (G, S, T, c) is a function $f : E(G) \to [0, +\infty)$ such that the following two conditions are satisfied:

- $f(e) \leq c(e)$ for all $e \in E(G)$;
- for all $v \in V(G) \setminus (S \cup T)$, we have $\sum\limits_{(x,v) \in E(G)} f(x, v) = \sum\limits_{(v,y) \in E(G)} f(v, y)$.

[1] We think of S as the set of sources, and of T as the set of sinks.
[2] A concrete example of a multi-source, multi-sink network is given in Problem 3.

The *value* of a flow f in (G, S, T, c) is

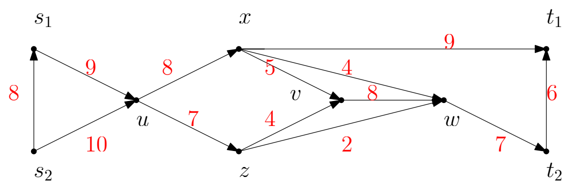
$$val(f) = f\Big(S, V(G) \setminus S\Big) - f\Big(V(G) \setminus S, S\Big),$$

where as usual, $f\Big(S, V(G) \setminus S\Big)$ is the sum of flows through the edges from S to $V(G) \setminus S$, and $f\Big(V(G) \setminus S, S\Big)$ is the sum of flows through the edges from $V(G) \setminus S$ to S. A *maximum flow* in (G, S, T, c) is a flow of maximum value, i.e. a flow f^* in (G, S, T, c) that satisfies the property that for all flows f in (G, S, T, c), we have that $val(f) \leq val(f^*)$.

Problem 2 (25 points). *Let (G, S, T, c) be a multi-source, multi-sink network. Construct a single-source, single-sink network (G^*, s^*, t^*, c^*) such that the maximum value of a flow in (G, S, T, c) is equal to the maximum value of a flow in (G^*, s^*, t^*, c^*). Make sure you prove that your construction is correct.*

> ***Hint:*** *Add two new vertices, call them s^* and t^*, to G, and connect s^* to all the sources and t^* to all the sinks. What about capacities?*

Problem 3 (25 points). *Using the transformation that you constructed in Problem 2, find a maximum flow in the multi-source, multi-sink network below ($S := \{s_1, s_2\}$ is the set of sources, $T := \{t_1, t_2\}$ is the set of sinks, and capacities are in red).*



Problem 4 (25 points). *Seven types of chemical (call them $C_1, \ldots, C_7$) are to be shipped in five trucks (call them $T_1, \ldots, T_5$). There are three containers storing each type of chemical, and the capacities of the trucks $T_1, \ldots, T_5$ are 6, 5, 4, 4, 3, respectively. For security reasons, no truck can carry more than one container of the same chemical. Determine whether it is possible to ship all 21 containers in the five trucks.*

> ***Hint:*** *You should probably start by formulating this as a network flow problem. Then either find a suitable flow, or find a cut of small capacity.*