NDMI011: Combinatorics and Graph Theory 1
HW#4

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due Monday, November 2, 2020 before midnight (Prague time)

Remark: Please e-mail me (ipenev@iuuk.mff.cuni.cz) your HW as a **PDF attachment** (no other format will be accepted).

Problem 1 (60 points)**.** *Let $n \geq 2$ be an integer, and let $(X, \mathcal{P})$ be a set system*[1] *such that $|X| = |\mathcal{P}| = n^2 + n + 1$. Assume that $|P| = n + 1$ for all $P \in \mathcal{P}$, and assume furthermore that all distinct $P_1, P_2 \in \mathcal{P}$ satisfy $|P_1 \cap P_2| \leq 1$. In what follows, we refer to the elements of X as* points *and to the elements of $\mathcal{P}$ as* lines*.*

(a) [12 points] Prove that for all distinct points $x_1, x_2 \in X$, there exists a unique line $P \in \mathcal{P}$ such that $x_1, x_2 \in P$.

> ***Hint:*** *Uniqueness should be easy (details?). For existence, start by noticing that for each line P, there are $\binom{n+1}{2}$ unordered pairs of distinct points of P. And how many unordered pairs of distinct points of X are there? Now what?*

(b) [12 points] Prove that each point is contained in at most $n + 1$ lines.

> ***Hint:*** *Show that otherwise, you get "too many" points in X.*

(c) [12 points] Prove that each point is contained in precisely $n + 1$ lines.

> ***Hint:*** *Consider the incidence graph of the set system $(X, \mathcal{P})$, and then count the number of edges in two ways: first using the degrees of points, and then using the degrees of lines. Use part (b).*

[1] So, X is a set and $\mathcal{P} \subseteq \mathscr{P}(X)$, where $\mathscr{P}(X)$ is the power set (i.e. the set of all subsets) of X.

(d) *[12 points] Prove that every two lines in $\mathcal{P}$ intersect, that is, prove that for all $P_1, P_2 \in \mathcal{P}$, we have that $P_1 \cap P_2 \neq \emptyset$.*

> ***Hint:*** *Fix an arbitrary line $P \in \mathcal{P}$. How many lines is it intersected by? (You should use one of the previous parts here.)*

(e) *[12 points] Prove that $(X, \mathcal{P})$ is a finite projective plane of order n.*

> ***Hint:*** *You might want to use one of the problems from HW#3, plus the previous parts.*

In each part, you may use the statements of all the preceding parts even if you did not prove them.

Problem 2 (40 points)**.** *Let X be the set of points and $\mathcal{P}$ the set of lines constructed in the proof of Theorem 2.3 from Lecture Notes 5. Using Problem 1 (or in some other way), show that $(X, \mathcal{P})$ is indeed a finite projective plane of order n. Make sure you carefully indicate where you are using the fact that $L_1, \dots, L_{n-1}$ are Latin squares, and where you are using orthogonality.*