NDMI011: Combinatorics and Graph Theory 1
HW#3

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due Monday, October 26, 2020 before midnight (Prague time)

Remark: Please e-mail me (ipenev@iuuk.mff.cuni.cz) your HW as a **PDF attachment** (no other format will be accepted).

Problem 1 (30 points)**.** *Let $n \geq 4$ be an integer, and let $X = \{0, 1, \dots, n\}$. Construct a set $\mathcal{P} \subseteq \mathscr{P}(X)$ such that*

- *$(X, \mathcal{P})$ satisfies (P1) and (P2) from the definition of a finite projective plane,*
- *$(X, \mathcal{P})$ does **not** satisfy (P0) from the definition of a finite projective plane, and*
- *every $P \in \mathcal{P}$ satisfies $|P| \geq 2$.*

Make sure you prove that your set system $(X, \mathcal{P})$ has the desired properties.

Problem 2 (35 points)**.** *Let $(X, \mathcal{P})$ be a set system such that X is finite, and all the following are satisfied:*[1]

(P0') there exist distinct $P_1, P_2 \in \mathcal{P}$ such that $|P_1|, |P_2| \geq 3$;

(P1) all distinct $P_1, P_2 \in \mathcal{P}$ satisfy $|P_1 \cap P_2| = 1$;

(P2) for all distinct $x_1, x_2 \in X$, there exists a unique $P \in \mathcal{P}$ such that $x_1, x_2 \in P$.

Prove that $(X, \mathcal{P})$ is a finite projective plane.

[1] Note that (P1) and (P2) are exactly as in the definition of a finite projective plane, but we have (P0') instead of (P0).

Problem 3 (35 points). *Prove Theorem 1.4(c) from Lecture Notes 4 directly, i.e.* ***without*** *using duality. The theorem is stated below, for your reference. (You may use parts (a) and/or (b) of Theorem 1.4 if you find them useful.)*

Theorem 1.4 from Lecture Notes 4. *Let* $(X, \mathcal{P})$ *be a finite projective plane of order* n*. Then all the following hold:*

(a) for each point $x \in X$*, exactly* $n+1$ *lines in* $\mathcal{P}$ *pass through* x*;*

(b) $|X| = n^2 + n + 1$*;*

(c) $|\mathcal{P}| = n^2 + n + 1$.