NDMI011: Combinatorics and Graph Theory 1
HW#2

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due Monday, October 19, 2020 before midnight (Prague time)

Remark: Please e-mail me (ipenev@iuuk.mff.cuni.cz) your HW as a **PDF attachment** (no other format will be accepted).

Problem 1 (40 points)**.** *Find the generating functions of the sequences below. Your final answers should **not** contain any infinite sums.*[1]

(a) $0, 0, 0, 0, -6, 6, -6, 6, -6, 6, -6, \ldots$

(b) $1, 0, 1, 0, 1, 0, \ldots$

(c) $1, 2, 1, 4, 1, 8, \ldots$

(d) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

Problem 2 (30 points)**.** *An* above-diagonal walk *on the $n \times n$ chessboard ($n \geq 0$) is a walk that satisfies all the following (see Figure 1):*

1. *it starts at the bottom left corner,*
2. *it ends at the upper right corner,*
3. *it never goes below the main diagonal,*
4. *it moves either one unit up or one unit to the right at each step.*

Show that for each integer $n \geq 0$, the number of above-diagonal walks on the $n \times n$ chessboard is equal to the number of binary trees on n nodes.[2]

[1] For example, if you were asked to find the generating function of $1, 1, 1, 1, \ldots$, then your final answer would be $\frac{1}{1-x}$, and **not** $\sum\limits_{n=0}^{\infty} x^n$.

[2] If $n = 0$, then the chessboard consists of only one point, which we consider to simultaneously be both the lower left and the upper right corner. So, there is exactly one above-diagonal walk on the 0×0 chessboard, and that walk is the 0-step walk.

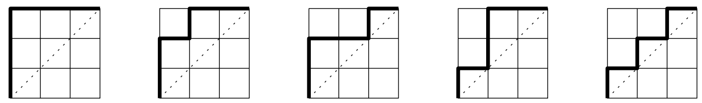

Figure 1: The five above-diagonal walks on the 3×3 chessboard.

> ***Hint:*** *For $n \geq 1$, break up your walk into two subwalks: the first subwalk should be between the lower left corner and the first point (other than the lower left corner) where the walk touches the main diagonal, and second subwalk should be from that point on the diagonal to the upper right corner of the chessboard.[3] Now what?*
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> ***Remark:*** *We showed in class that the number of binary trees on n nodes ($n \geq 0$) is $\frac{1}{n+1}\binom{2n}{n}$. So, this problem implies that the number of above-diagonal walks on the $n \times n$ chessboard ($n \geq 0$) is precisely $\frac{1}{n+1}\binom{2n}{n}$, i.e. the n-th Catalan number.*

Problem 3 (30 points)**.** *Consider the following infinite random walk on the integer line $\mathbb{Z}$: we start at 1, and at each step, we move either one unit to the left (-1) or one unit to the right ($+1$) at random (i.e. we make each choice with probability $\frac{1}{2}$, and choices at different steps are independent). Compute the probability P that we reach the origin at some point during our walk.*

> ***Hint:*** *This is similar to (but easier than) the calculation from section 3 of Lecture Notes 3. For all non-negative integers n,*
>
> - *let a_n be the number of n-step random walks starting at 1 (and following our rules), ending the origin, and never passing though the origin until the very end of the walk;*
> - *let b_n be the number of n-step random walks starting at 2 (and following our rules), ending the origin, and never passing though the origin until the very end of the walk;*
> - *let $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating functions of $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, respectively.*
>
> *Now proceed similarly as in the Lecture Notes.[4]*

[3]It is possible that the second subwalk is a zero-step walk, but the first one cannot be.

[4]Unlike in the Lecture Notes, though, you should not need to consider random walks starting at 3.