Isometric and induced path partitions: a new upper bound and a characterization of some extremal graphs

Irena Penev[*] R. B. Sandeep[†] D.K. Supraja[‡] S Taruni[§]

Abstract

An *isometric path* is a shortest path between two vertices. An *isometric path partition* (IPP) of a graph G is a set $\mathcal{I}$ of vertex-disjoint isometric paths in G that partition the vertices of G. The *isometric path partition number* of G, denoted by $\text{ipp}(G)$, is the minimum cardinality of an IPP of G. An *induced path partition* (IndPP) of a graph G is a set $\mathcal{I}$ of vertex-disjoint induced paths in G that partition the vertices of G. The *induced path partition number* of G, denoted by $\text{indpp}(G)$, is the minimum cardinality of an IndPP of G. In this article, we study both these parameters and observe that every graph G satisfies $\text{indpp}(G) \leq \text{ipp}(G) \leq |V(G)| - \nu(G)$, where $\nu(G)$ is matching number of G. We further prove that a connected graph G is extremal with respect to this upper bound, i.e. satisfies $\text{ipp}(G) = |V(G)| - \nu(G)$, (resp. $\text{indpp}(G) = |V(G)| - \nu(G)$), if and only if either (i) all blocks of G are odd complete graphs, or (ii) all blocks of G except one are odd complete graphs, and the unique block B of G that is not an odd complete graph is even and satisfies $\text{ipp}(B) = |V(B)| - \nu(B)$ (resp. $\text{indpp}(B) = |V(B)| - \nu(B)$). As corollaries of this result, we obtain a full structural characterization of all connected odd graphs that are extremal with respect to our upper bound, as well as of all extremal block graphs.

1 Introduction

In what follows, all graphs are finite, simple, and nonnull. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively, and we say that the graph G is *even* (resp. *odd*) if $|V(G)|$ is even (resp. odd). An *n-vertex graph* is a graph that contains exactly n vertices.

Graph partitioning and covering problems are central topics in graph theory and algorithms, including problems such as dominating set (covering by stars), clique covering (covering by cliques), coloring (partitioning by independent sets), and covering/partitioning by paths or cycles. In particular, the problems of partitioning and covering by paths have their connections to important graph-theoretic results such as the Gallai-Milgram theorem [10], Berge's path partition conjecture [2], and also to the celebrated Graph Minor project [13, 20]. These problems also have applications in various domains, including artificial intelligence [12], bioinformatics [14], code optimization [3], program testing [18], to name a few. In recent years, various types of path partitions have been studied, such as unrestricted path partition [6, 11, 21], induced path partition [5, 15, 19], and isometric (i.e. shortest) path partition [4, 7]. See [7, 16] for a broader overview of path partition and related problems in the literature.

The focus of this article is on the problem of partitioning the vertex set of a graph into isometric paths. Let us be more precise. A *path* (resp. *induced path*) in a graph G is a subgraph (resp. induced subgraph) P of G such that P is a path. The *length* of a path P, denoted by $\text{length}(P)$, is the number of edges that it contains. Given vertices x and y in a graph G, a *shortest* or *isometric* path between x and y in G is a path in G whose endpoints are x and y, and is of smallest possible length among all paths

---

[*] Computer Science Institute (IÚUK, MFF), Charles University, Prague, Czech Republic. Supported by GAČR grant 25-17377S and by MSCA-RISE-2020-101007705 project *RandNET*. Email: ipenev@iuuk.mff.cuni.cz.

[†] Department of Computer Science and Engineering, Indian Institute of Technology Dharwad, India. Supported by ANRF (SERB) grant MTR/2022/000692. Email: sandeeprb@iitdh.ac.in.

[‡] Indian Institute of Technology Dharwad, India. Email: dksupraja95@gmail.com.

[§] Centro de Modelamiento Matemático (CNRS IRL2807), Universidad de Chile, Santiago, Chile. Supported by Centro de Modelamiento Matemático (CMM) BASAL fund FB210005 for center of excellence from ANID-Chile and by MSCA-RISE-2020-101007705 project *RandNET*. Email: tsridhar@cmm.uchile.cl.

between x and y in G. When we say that "P is an isometric path in G," we mean that P is a shortest path between its endpoints in G (here, it is possible that P has only one vertex). We now state the main definitions.

Definition 1.1. *An* induced path partition *(or* IndPP *for short) of a graph G is a collection of induced paths in G whose vertex sets partition $V(G)$. The* induced path partition number *of G, denoted by indpp(G), is the smallest cardinality of any IndPP of G. A* minimum *IndPP of G is an IndPP of G of cardinality indpp(G).*

Definition 1.2. *An* isometric path partition *(or* IPP *for short) of a graph G is a collection of isometric paths in G whose vertex sets partition $V(G)$. The* isometric path partition number *of G, denoted by ipp(G), is the smallest cardinality of any IPP of G. A* minimum *IPP of G is an IPP of G of cardinality ipp(G).*

Clearly, every isometric path in G is an induced path of G, but the converse is not true in general, which implies that $indpp(G) \leq ipp(G)$. We note, however, that any induced path of length at most two in G is an isometric path of G, a fact that we will repeatedly use in this paper.

The *distance* between vertices x and y in a graph G, denoted by $\text{dist}_G(x, y)$, is defined as the length of a shortest path between x and y (if no such path exists, i.e. if x and y belong to different connected components of G, then we define $\text{dist}_G(x, y) := \infty$); if $x = y$, then clearly, $\text{dist}_G(x, y) = 0$. The *diameter* of a graph G is defined as $\text{diam}(G) := \max\big\{\text{dist}_G(x, y) \mid x, y \in V(G)\big\}$. Clearly, any isometric path in G is of length at most $\text{diam}(G)$ (i.e. contains at most $\text{diam}(G) + 1$ many vertices), and consequently, as pointed out in [17], G satisfies

$$\text{ipp}(G) \quad \geq \quad \left\lceil \tfrac{|V(G)|}{\text{diam}(G)+1} \right\rceil.$$

In this article, we give an upper bound for the IndPP and IPP numbers in terms of the matching number, and we characterize the connected IPP-extremal (resp. IndPP-extremal) graphs (i.e. those graphs whose IPP (resp. IndPP) number is equal to our upper bound) in terms of their blocks. These results are described in subsection 1.2 (see Proposition 1.3, Theorem 1.6 and Theorem 1.7). As a corollary of our main result (Theorem 1.6), we give a full structural characterization of the connected odd IndPP-extremal and IPP-extremal graphs (see Corollary 3.2), as well of the IndPP-extremal and IPP-extremal block graphs (see Corollary 3.5).

The remainder of the paper is organized as follows. In subsection 1.1, we provide the context for our work. In subsection 1.2, we describe our main results. In subsection 1.3, we introduce some (mostly standard) terminology and notation that we will use throughout the paper. In section 2, we prove Theorem 1.6 (our main theorem). In section 3, we derive some corollaries of Theorem 1.6. Finally, in section 4, we propose some open questions.

1.1 Previous work

For a set X, a collection $\mathcal{Z}$ of subsets of X is said to *cover* X if for all $x \in X$, there exists some $Z \in \mathcal{Z}$ such that $x \in Z$. An *isometric path cover* (or *IPC* for short) of a graph G is a collection of isometric paths of G whose vertex sets cover $V(G)$. The *isometric path cover number* of G, denoted by $\text{ipc}(G)$, is the minimum cardinality of any IPC of G. IPC was introduced by Fitzpatrick [8], and it is obviously related to IPP. Indeed, any IPP of a graph G is, in particular, an IPC of G, and it follows that $\text{ipp}(G) \geq \text{ipc}(G)$.

Fitzpatrick referred to the IPC number by different names such as the *precinct number* [8] or the *isometric path number* [9]. This concept proved useful in a particular variation of the game of cops and robbers. For any graph G where a single cop suffices to capture the robber, a strategy for the cop can be determined by projecting the game positions onto isometric paths within the graph. Fitzpatrick [8] observed that all graphs G satisfy $\text{ipc}(G) \geq \big\lceil |V(G)|/(\text{diam}(G) + 1)\big\rceil$, and he described some classes of graphs that reach this lower bound (i.e. graphs for which this inequality becomes an equality). In particular, Fitzpatrick et al. [9] proved that the hypercube Q_n satisfies $\text{ipc}(Q_n) \geq 2^n/(n + 1)$, and that this lower bound is in fact tight when $n + 1$ is a power of 2.

The exact values of the IPP number have been established for hypercubes Q_n when $n+1$ is a power of 2, as well as for $r \times r$ tori [17]. We remark that the results of [17] relied on previously established results for the IPC number, as well as on the above-mentioned fact that every graph G satisfies $\text{ipp}(G) \geq \text{ipc}(G)$. To our knowledge, no other significant structural results have been obtained for either IPP or IndPP.

However, there have been a number of algorithmic results for these two parameters. In particular, Le et al. [15] proved that it is NP-complete to decide whether or not the vertex set of a connected graph can be partitioned into two subsets each of which induces a path; consequently, computing the IndPP number of a graph is NP-hard. Similarly, Manuel [17] showed that computing an optimal IPP is NP-hard on undirected graphs; this was extended in [7] to prove that IPP (studied under the name *shortest path partition*, SPP in [7]) is NP-hard even when restricted to bipartite graphs that are sparse (in particular, when they have degeneracy at most 5). It was further shown in [7] that IndPP and IPP belong to FPT when parameterized by standard structural parameters such as the vertex cover and the neighborhood diversity (this result was obtained using Integer Linear Programming), and moreover, that IPP can be solved in XP-time on undirected graphs when parameterized by solution size. IPP is known to be NP-hard on split graphs while the problem is polynomial-time solvable for cographs and chain graphs [4]. Additionally, polynomial-time algorithms are known for computing optimal (unrestricted) path partitions in trees [21] and for computing optimal induced path partitions in graphs with special blocks [19]. Since, for these graph classes, the (unrestricted) path partition and induced path partition coincide with the isometric path partition, this immediately yields polynomial-time algorithms for computing a minimum IPP and IndPP in trees and graphs with special blocks [19, 21].

1.2 Our contribution

A *matching* of a graph G is a set of edges of G, no two of which share an endpoint. The *matching number* of G, denoted by $\nu(G)$, is the largest cardinality of any matching in G; clearly, $|V(G)| \geq 2\nu(G)$. A *maximum matching* of G is a matching of G of cardinality $\nu(G)$. For a vertex v and a matching M of G, we say that v is *M-saturated* if v is incident with some edge in M, and otherwise, we say that v is *M-unsaturated*. A *perfect matching* of G is a matching M of G such that every vertex of G is M-saturated; clearly, G has a perfect matching if and only if $|V(G)| = 2\nu(G)$.

For notational convenience, we will identify any one-vertex path with its unique vertex, and we will identify any two-vertex path with its unique edge. Clearly, any one- or two-vertex path is isometric. So, if M is a matching of a graph G, and U is the set of all M-unsaturated vertices of G, then $\mathcal{I} := M \cup U$ is an IPP of G, and we see that $\text{ipp}(G) \leq |\mathcal{I}| = |M| + |U| = |M| + \big(|V(G)| - 2|M|\big) = |V(G)| - |M|$. By choosing M to be a maximum matching of G, so that $|M| = \nu(G)$, we obtain the following proposition.

Proposition 1.3. *Every graph G satisfies $\text{indpp}(G) \leq \text{ipp}(G) \leq |V(G)| - \nu(G)$.*

Let us say that a graph G is *IPP-extremal* if it satisfies $\text{ipp}(G) = |V(G)| - \nu(G)$ (so that the inequality from Proposition 1.3 becomes an equality). Similarly, let us say that a graph G is *IndPP-extremal* if it satisfies $\text{indpp}(G) = |V(G)| - \nu(G)$.

Proposition 1.4. *If a graph G is IndPP-extremal, then both the following hold:*

- *G is IPP-extremal;*
- *every minimum IPP of G is also a minimum IndPP of G.*

Proof. Fix an IndPP-extremal graph G, so that $\text{indpp}(G) = |V(G)| - \nu(G)$. Combined with Proposition 1.3, this implies that $|V(G)| - \nu(G) = \text{indpp}(G) \leq \text{ipp}(G) \leq |V(G)| - \nu(G)$, and consequently, $\text{indpp}(G) = \text{ipp}(G) = |V(G)| - \nu(G)$. Thus, G is IPP-extremal. Since every IPP of G is an IndPP of G, it further follows that any minimum IPP of G is a minimum IndPP of G. □

We note that the converse of Proposition 1.4 is false: not all IPP-extremal graphs are IndPP-extremal. For example, it is easy to check that the wheel on six vertices represented in Figure 1 is IPP-extremal, but not IndPP-extremal.

The goal of this paper is to give a partial characterization of IPP-extremal graphs and IndPP-extremal graphs. The following proposition reduces this problem to the connected case.

Proposition 1.5. *Let G be a graph. Then both the following hold:*

(a) G is IPP-extremal if and only if all components of G are IPP-extremal;

(b) G is IndPP-extremal if and only if all components of G are IndPP-extremal.

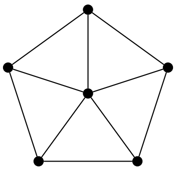

Figure 1: A graph that is IPP-extremal, but not IndPP-extremal.

Proof. We prove (a); the proof of (b) is completely analogous. Let $G_1, \dots, G_t$ be the connected components of G. For each $i \in \{1, \dots, t\}$, let M_i be a maximum matching of G_i, and let $\mathcal{I}_i$ be a minimum IPP of G. Then $M := M_1 \cup \dots \cup M_t$ is a maximum matching of G, and $\mathcal{I} := \mathcal{I}_1 \cup \dots \cup \mathcal{I}_t$ is minimum IPP of G. We now have that

$$\begin{array}{rcl} \text{ipp}(G) & = & \sum\limits_{i=1}^{t} \text{ipp}(G_i) \\ & \overset{(*)}{\leq} & \sum\limits_{i=1}^{t} \Big(|V(G_i)| - \nu(G_i)\Big) \\ & = & \Big(\sum\limits_{i=1}^{t} |V(G_i)|\Big) - \Big(\sum\limits_{i=1}^{t} \nu(G_i)\Big) \\ & = & |V(G)| - \nu(G), \end{array}$$

where (*) follows from Proposition 1.3. Moreover, the inequality (*) is an equality precisely when $\text{ipp}(G_i) = |V(G_i)| - \nu(G_i)$ for all $i \in \{1, \dots, t\}$. This proves (a). $\square$

A *block* of a graph is a maximal biconnected induced subgraph of that graph. Our main results are the following.

Theorem 1.6. *Let G be a connected graph. Then G is IPP-extremal if and only if G satisfies one of the following:*

- *(i) all blocks of G are odd complete graphs;*
- *(ii) all blocks of G except one are odd complete graphs, and the unique block of G that is not an odd complete graph is even and IPP-extremal.*

Theorem 1.7. *Let G be a connected graph. Then G is IndPP-extremal if and only if G satisfies one of the following:*

- *(i) all blocks of G are odd complete graphs;*
- *(ii) all blocks of G except one are odd complete graphs, and the unique block of G that is not an odd complete graph is even and IndPP-extremal.*

1.3 Terminology and notation

When we write "$P = p_1 \dots p_t$ is a path" ($t \geq 1$) we mean that P is a t-vertex graph with $V(P) = \{p_1, \dots, p_t\}$ and $E(P) = \{p_i p_{i+1} \mid 1 \leq i \leq t-1\}$; under these circumstances, we say that p_1 and p_t are the *endpoints* of the path P (or that P is a path *between* p_1 and p_t), whereas $p_2, \dots, p_{t-1}$ are the *internal vertices* of P (if $t = 1$, then p_1 is the only endpoint of P, and P has no internal vertices).

For a vertex v in a graph G, the *open neighborhood* of v in G, denoted by $N_G(v)$, is the set of all neighbors of v in G, whereas the *closed neighborhood* of v in G is the set $N_G[v] = \{v\} \cup N_G(v)$. For a vertex $v \in V(G)$ and a set $S \subseteq V(G) \setminus \{v\}$, we say that v is *complete* (resp. *anticomplete*) to S in G if v is adjacent (resp. nonadjacent) to all vertices of S, and we say that v is *mixed* on S in G if v is neither complete nor anticomplete to S in G (i.e. if v has both a neighbor and a nonneighbor in S). Recall that, formally, an edge is simply a set of two distinct vertices (the endpoints of the edge). So, it makes sense to speak of a vertex being (anti)complete to or mixed on an edge (as long as the vertex is not incident with the edge). For disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (resp. *anticomplete*) to Y in G if every vertex in X is complete (resp. anticomplete) to Y in G.

A *clique* (resp. *stable set*) in a graph G is any set of pairwise adjacent (resp. nonadjacent) vertices of G. Note that $\emptyset$ is both a clique and a stable set in any graph. A clique or a stable set is *even* (resp. *odd*) if its cardinality is even (resp. odd). A vertex v of a graph G is *simplicial* if $N_G(v)$ is a (possibly empty) clique of G.

For a graph G and a nonempty set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. For $S \subsetneqq V(G)$, we denote by $G \setminus S$ the subgraph of G obtained by deleting all vertices of S, that is, $G \setminus S := G[V(G) \setminus S]$; if $S = \{v\}$, we may write $G \setminus v$ instead of $G \setminus S$ or $G \setminus \{v\}$. For a graph G, a *cutset* of G is any set $S \subsetneqq V(G)$ such that $G \setminus S$ contains more connected components than G does. For a graph G on at least two vertices, a *cut-vertex* of G is a vertex v of G such that $G \setminus v$ has more connected components than G has (i.e. such that $\{v\}$ is a cutset of G).

A graph G is *biconnected* if it is connected and contains no cut-vertices. Note that all complete graphs are biconnected. A *block* of a graph G is a maximal biconnected induced subgraph of G. Consistently with our terminology above, a block B of a graph G is *even* (resp. *odd*) if $|V(B)|$ is even (resp. odd). Clearly, every connected graph G that is not biconnected contains a cut-vertex. A *block graph* is a graph, all of whose blocks are complete graphs. A *leaf-block* of a graph G is a block of G that contains exactly one cut-vertex of G. Note that any connected graph that is not biconnected contains at least two leaf-blocks. A *leaf-clique* of a graph G is a nonempty clique C of G such that $G[C]$ is a leaf-block of G.

2 Proof of Theorems 1.6 and 1.7

Proposition 2.1. *Let G be a connected IPP-extremal graph, and let M be a maximum matching of G. Then no M-unsaturated vertex of G is mixed on any edge of M. Moreover, G contains at most one M-unsaturated vertex.*

Proof. Let U be the set of all M-unsaturated vertices of G. We must show that no vertex in U is mixed on any edge in M, and that $|U| \leq 1$. Since M is a maximum matching of G, we have that $|M| = \nu(G)$, and consequently, $|U| = |V(G)| - 2|M| = |V(G)| - 2\nu(G)$. So, $\mathcal{I} := M \cup U$ is an IPP of G of size $|V(G)| - \nu(G)$; since $\text{ipp}(G) = |V(G)| - \nu(G)$, it follows that $\mathcal{I}$ is in fact a minimum IPP of G.

First, suppose toward a contradiction that some M-unsaturated vertex u is mixed on an edge $xy \in M$. By symmetry, we may assume that $ux \in E(G)$ and $uy \notin E(G)$. But now, $\big(M \setminus \{xy\}\big) \cup \big(U \setminus \{u\}\big) \cup \{uxy\}$ is an IPP of G of size $\big(|M| - 1\big) + \big(|U| - 1\big) + 1 = |V(G)| - \nu(G) - 1$, contrary to the assumption that G is IPP-extremal and therefore satisfies $\text{ipp}(G) = |V(G)| - \nu(G)$.

It remains to show that $|U| \leq 1$. Suppose otherwise, and fix distinct $u_1, u_2 \in U$. Let P' be a shortest path between u_1 and u_2 in G. But then

$$\mathcal{I}' \quad := \quad \{P'\} \cup \big\{P \in \mathcal{I} \mid V(P') \cap V(P) = \emptyset\big\} \cup \big\{v \in V(G) \setminus V(P') \mid \exists v' \in V(P') \text{ s.t. } vv' \in M\big\}$$

is an IPP of G size at most $|\mathcal{I}| - 1$, contrary to the minimality of $\mathcal{I}$. □

Proposition 2.2. *Let G be a connected even IPP-extremal graph. Then $|V(G)| = 2\nu(G)$ and $\text{ipp}(G) = \nu(G)$. Moreover, G admits a perfect matching, and any perfect matching of G is a minimum IPP of G.*

Proof. Fix a maximum matching M of G. By Proposition 2.1, at most one vertex of G is M-unsaturated. But since G is even, this implies that no vertex of G is M-unsaturated, that is, M is a perfect matching of G. Consequently, $|V(G)| = 2\nu(G)$. Since $\text{ipp}(G) = |V(G)| - \nu(G)$, it follows that $\text{ipp}(G) = \nu(G)$. Finally, since any perfect matching is an IPP of G of size $\nu(G)$, it follows that any perfect matching of G is a minimum IPP of G. □

Proposition 2.3. *Let G be a connected odd IPP-extremal graph. Then $|V(G)| = 2\nu(G)+1$ and $\text{ipp}(G) = \nu(G) + 1$. Moreover, for every vertex $u \in V(G)$, the graph $G \setminus u$ admits a perfect matching, and furthermore, for any perfect matching M_u of $G \setminus u$, we have that $M_u \cup \{u\}$ is a minimum IPP of G.*

Proof. Since G is odd, Proposition 2.1 guarantees that for every maximum matching M of G, exactly one vertex of G is M-unsaturated; consequently, $|V(G)| = 2\nu(G) + 1$. Since $\text{ipp}(G) = |V(G)| - \nu(G)$, it follows that $\text{ipp}(G) = \nu(G) + 1$.

It is clear that if M_u is a perfect matching of $G \setminus u$ for some vertex u, then M_u is a maximum matching of G, and that $M_u \cup \{u\}$ is an IPP of G; since $\text{ipp}(G) = \nu(G) + 1$, we see that $M_u \cup \{u\}$ is in fact a minimum IPP of G.

It remains to prove that for all $u \in V(G)$, the graph $G \setminus u$ admits a perfect matching. Set

$$\mathcal{U} \quad := \quad \{u \in V(G) \mid G \setminus u \text{ admits a perfect matching}\}.$$

We must show that $\mathcal{U} = V(G)$. Suppose otherwise, so that $V(G) \setminus \mathcal{U} \neq \emptyset$. Since $|V(G)| = 2\nu(G) + 1$, it is clear that $\mathcal{U} \neq \emptyset$.[1] Since G is connected, there exist adjacent vertices $u \in \mathcal{U}$ and $v \in V(G) \setminus \mathcal{U}$. Since $u \in \mathcal{U}$, $G \setminus u$ admits a perfect matching, call it M_u. Clearly, M_u is a maximum matching of G, and u is the only M_u-unsaturated vertex of G. Then v is M_u-saturated, and therefore, there exists some $v' \in V(G) \setminus \{u, v\}$ such that $vv' \in M_u$. Since u is M_u-unsaturated and adjacent to v, Proposition 2.1 guarantees that u is also adjacent to v'.[2] But now $M_v := \big(M_u \setminus \{vv'\}\big) \cup \{uv'\}$ is a perfect matching of $G \setminus v$, contrary to the fact that $v \notin \mathcal{U}$. □

Proposition 2.4. *Let C be a leaf-clique of a graph G, let v be the (unique) cut-vertex of G that belongs to C, and let x and y be distinct vertices in $C \setminus \{v\}$. Set $G' := G \setminus \{x, y\}$. Then both the following hold:*

(a) G is IPP-extremal if and only if G' is IPP-extremal.

(b) G is IndPP-extremal if and only if G' is IndPP-extremal.

Proof. We prove (a) and (b) simultaneously. We begin by observing all vertices $z \in C \setminus \{v\}$ satisfy $N_G[z] = C$, and consequently, all vertices in $C \setminus \{v\}$ are simplicial in G. In particular, we have that $N_G[x] = N_G[y] = C$, and that vertices x and y are simplicial in G.

Claim 2.4.1. *All the following hold:*

(1) $|V(G')| = |V(G)| - 2$;

(2) $\nu(G') = \nu(G) - 1$;

(3) $|V(G')| - \nu(G') = |V(G)| - \nu(G) - 1$.

Proof of Claim 2.4.1. Part (1) follows immediately from the construction, and part (3) follows immediately from (1) and (2). Thus, we just need to prove (2).

Clearly, if M' is a maximum matching of G', then $M' \cup \{xy\}$ is a matching of G, and so $\nu(G) \geq |M' \cup \{xy\}| = |M'| + 1 = \nu(G') + 1$, which implies that $\nu(G') \leq \nu(G) - 1$.

It remains to show that $\nu(G') \geq \nu(G) - 1$. Clearly, it is enough to show that G contains a maximum matching M that contains the edge xy, for then $M \setminus \{xy\}$ will be a matching of G', and consequently, $\nu(G') \geq |M \setminus \{xy\}| = |M| - 1 = \nu(G) - 1$, which is what we need.

Now, fix any maximum matching M of G. We may assume that $xy \notin M$, for otherwise we are done. It is clear that at least one of x, y is M-saturated, for otherwise, $M \cup \{xy\}$ would be a matching of G of size $|M| + 1$, contrary to the maximality of M. Suppose first that exactly one of x and y is M-saturated; by symmetry, we may assume that x is M saturated, while y is not. Then there exists some $x' \in V(G) \setminus \{x, y\}$ such that $xx' \in M$. But now $M' := (M \setminus \{xx'\}) \cup \{xy\}$ is a matching of size $|M| = \nu(G)$, and by construction, $xy \in M'$. We may now assume that both x and y are M-saturated. Since $xy \notin M$, we see that there exist distinct $x', y' \in V(G) \setminus \{x, y\}$ such that $xx', yy' \in M$. Since $N_G[x] = N_G[y] = C$, we see that $x', y' \in C$, and since C is a clique, we see that $x'y' \in E(G)$. But now $(M \setminus \{xx', yy'\}) \cup \{xy, x'y'\}$ is a matching of G of size $|M| = \nu(G)$, and by construction, $xy \in M$. This completes the proof of (2). ◆

Claim 2.4.2. *Neither x nor y is an internal vertex of any induced path in G. Consequently, neither x nor y is an internal vertex of any isometric path in G.*

---

[1] Indeed, if M is any maximum matching of G, then the fact that $|V(G)| = 2\nu(G) + 1$ implies that exactly one vertex of G, call it v_M, is M-unsaturated; but clearly, M is a perfect matching of $G \setminus v_M$, and so $v_M \in \mathcal{U}$.

[2] Otherwise, the M_u-unsaturated vertex u would be mixed on the matching edge $vv' \in M_u$, contrary to Proposition 2.1.

Proof of Claim 2.4.2. Since all isometric path are induced, it suffices to prove the first statement. Fix an induced path P in G. Then any internal vertex of P has two nonadjacent neighbors in P, and therefore in G as well (because the path P is induced). Since x and y are simplicial, it follows that neither x nor y is an internal vertex of P. ♦

Claim 2.4.3. *Any induced (resp. isometric) path of G' is also an induced (resp. isometric) path in G.*

Proof of Claim 2.4.3. Since G' is an induced subgraph of G, it is clear that all induced paths in G' are also induced paths in G. Now, fix an isometric path P of G', and let p and q be its endpoints (possibly $p = q$). We must show that the path P is isometric in G. Suppose otherwise, and fix an isometric path Q between p and q in G such that $\text{length}(Q) < \text{length}(P)$. Then at least one of x and y must be an internal vertex of Q, for otherwise, Q would be a path in G', contrary to the fact that the path P is isometric in G'. But this contradicts Claim 2.4.2. ♦

Claim 2.4.4. *Some minimum IPP of G contains the path xy. Similarly, some minimum IndPP of G contains the path xy.*

Proof of Claim 2.4.4. We prove the claim for IPP; the proof for IndPP is completely analogous.[3]

First of all, it is clear that xy is, in fact, an isometric path in G. Now, fix a minimum IPP $\mathcal{I}$ of G, and fix $P_x, P_y \in \mathcal{I}$ such that $x \in V(P_x)$ and $y \in V(P_y)$. Since the paths P_x and P_y are isometric paths in G, they are induced paths of G. Moreover, by Claim 2.4.2, x is an endpoint of P_x, and y is an endpoint of P_y.

Suppose first that $P_x = P_y$. Then x and y are the endpoints of $P_x = P_y$; since x and y are adjacent, and the path $P_x = P_y$ is isometric in G, we deduce that $P_x = P_y = xy$. Thus, $xy \in \mathcal{I}$, and we are done.

From now on, we assume that $P_x \neq P_y$. Suppose that at least one of P_x and P_y has length zero. By symmetry, we may assume that $\text{length}(P_x) = 0$. Now, if $\text{length}(P_y) = 0$, then $\big(\mathcal{I} \setminus \{P_x, P_y\}\big) \cup \{xy\}$ is an IPP of G of size $|\mathcal{I}| - 1$, contrary to the minimality of $\mathcal{I}$. So, $\text{length}(P_y) \geq 1$. Set $P_y = yp_1 \dots p_t$ ($t \geq 1$). But now $\big(\mathcal{I} \setminus \{P_x, P_y\}\big) \cup \{xy, p_1 \dots p_t\}$ is an IPP of G of size $|\mathcal{I}|$, and by construction, this IPP contains xy.

It remains to consider the case when both P_x and P_y are of length at least one. Set $P_x = xx_1 \dots x_s$ and $P_y = yy_1 \dots y_t$ ($s, t \geq 1$). Obviously, at most one of P_x and P_y contains the cut-vertex v; by symmetry, we may assume that $v \notin V(P_x)$. So, P_x is in fact an induced path of the complete graph $G[C \setminus \{v\}]$, and we deduce that $\text{length}(P_x) = 1$, i.e. $P_x = xx_1$, with $x_1 \in C \setminus \{v\}$. But now $\big(\mathcal{I} \setminus \{P_x, P_y\}\big) \cup \{xy, x_1y_1 \dots y_t\}$ is an IPP of G of size $|\mathcal{I}| = \text{ipp}(G)$, and by construction, this IPP contains xy. ♦

Claim 2.4.5. *$\text{ipp}(G') = \text{ipp}(G) - 1$ and $\text{indpp}(G') = \text{indpp}(G) - 1$.*

Proof of Claim 2.4.5. We prove the first statement; the proof of the second statement is completely analogous.

We first show that $\text{ipp}(G') \geq \text{ipp}(G) - 1$. Fix a minimum IPP $\mathcal{I}'$ of G'. By Claim 2.4.3, every path in $\mathcal{I}'$ is isometric in G. But now $\mathcal{I}' \cup \{xy\}$ is an IPP of G. We deduce that $\text{ipp}(G) \leq |\mathcal{I}' \cup \{xy\}| = |\mathcal{I}'| + 1 = \text{ipp}(G') + 1$, and consequently, $\text{ipp}(G') \geq \text{ipp}(G) - 1$.

It remains to show that $\text{ipp}(G') \leq \text{ipp}(G) - 1$. Using Claim 2.4.4, we fix a minimum IPP $\mathcal{I}$ of G such that $xy \in \mathcal{I}$. But then $\mathcal{I} \setminus \{xy\}$ is an IPP of G', and we deduce that $\text{ipp}(G') \leq |\mathcal{I} \setminus \{xy\}| = |\mathcal{I}| - 1 = \text{ipp}(G) - 1$. ♦

By Claim 2.4.1, we have that $|V(G')| - \nu(G') = |V(G)| - \nu(G) - 1$. So, by Claim 2.4.5, we have the following:

- $\text{ipp}(G') = |V(G')| - \nu(G')$ if and only if $\text{ipp}(G) = |V(G)| - \nu(G)$;
- $\text{indpp}(G') = |V(G')| - \nu(G')$ if and only if $\text{indpp}(G) = |V(G)| - \nu(G)$.

Thus, both (a) and (b) hold, and we are done. □

Proposition 2.5. *Let G be a graph that contains an odd leaf-clique C, and assume that v is the unique cut-vertex of G that belongs to C. Set $G' := G \setminus \big(C \setminus \{v\}\big)$. Then all the following hold:*

---

[3] Indeed, as the reader can check, our argument remains valid if we simply replace all instances of "IPP" by "IndPP," all instances of "ipp(G)" by "indpp(G)," and all instances of "isometric" by "induced".

(a) the blocks of G are precisely $G[C]$ and the blocks of G';

(b) G is IPP-extremal if and only if G' is IPP-extremal;

(c) G is IndPP-extremal if and only if G' is IndPP-extremal.

Proof. Part (a) is obvious, and we just need to prove (b) and (c). Since C is an odd leaf-clique of G, we see that $|C| \geq 3$, and that $C \setminus \{v\}$ is a nonempty even clique. Set $k := \frac{|C \setminus \{v\}|}{2}$ and $C \setminus \{v\} = \{x_1, y_1, \dots, x_k, y_k\}$. So, we can obtain G' from G by iteratively deleting the sets $\{x_1, y_1\}, \dots, \{x_k, y_k\}$ from G. Now (b) and (c) follow simply by applying Proposition 2.4 k times. $\square$

A graph is *chordal* if all induced cycles in it are triangles. The *diamond* is the graph on four vertices obtained from the complete graph K_4 by deleting one edge. A graph is *diamond-free* if none of its induced subgraphs is (isomorphic to) the diamond.

Theorem 2.6. *[1] Block graphs are precisely the diamond-free chordal graphs.*

Proposition 2.7. *Let G be a connected odd IPP-extremal graph. Then G is a block graph. Moreover, if G is biconnected, then G is an odd complete graph.*

Proof. Clearly, biconnected block graphs are precisely the complete graphs. So, the second statement follows from the first (and from the assumption that G is odd). Thus, it remains to show that G is a block graph. In view of Theorem 2.6, it suffices to show that G is a diamond-free chordal graph. By Proposition 2.3, we have that $\text{ipp}(G) = \nu(G) + 1$.

Claim 2.7.1. *G is diamond-free.*

Proof of Claim 2.7.1. Suppose otherwise, and fix an induced diamond D in G, where $V(D) = \{a, b, c, d\}$ and $E(D) = \{ab, bc, cd, da, bd\}$. Using Proposition 2.3, we fix a perfect matching M of $G \setminus d$. Then M is a maximum matching in G such that d is the unique M-unsaturated vertex of G, and in particular, a, b, and c are all M-saturated. Clearly, $M \cup \{d\}$ is an IPP of G. Since d is M-unsaturated, we see that $E(D) \cap M \subseteq \{ab, bc\}$. But since edges ab and bc share an endpoint, the matching M contains at most one of them. So, by symmetry, we may assume that either $E(D) \cap M = \{ab\}$ or $E(D) \cap M = \emptyset$. In each case, we will derive a contradiction by exhibiting an IPP of G of size $\nu(G)$, contrary to the fact that $\text{ipp}(G) = \nu(G) + 1$.

Suppose first that $E(D) \cap M = \{ab\}$. Then, since c is M-saturated, there exists some $c' \in V(G) \setminus V(D)$ such that $cc' \in M$. Since d is M-unsaturated and adjacent to c, Proposition 2.1 guarantees that $dc' \in E(G)$. But then $\big(M \setminus \{ab, cc'\}\big) \cup \{abc, dc'\}$ is an IPP of G of size $|M| = \nu(G)$, a contradiction.

Suppose now that $E(D) \cap M = \emptyset$. Then, since a, b, and c are M-saturated, there exist pairwise distinct vertices $a', b', c' \in V(G) \setminus V(D)$ such that $aa', bb', cc' \in M$. Since d is M-unsaturated and adjacent to a, b, and c, Proposition 2.1 guarantees that $da', db', dc' \in E(G)$. If $a'b' \notin E(G)$, then $\big(M \setminus \{aa', bb'\}\big) \cup \{a'db', ab\}$ is an IPP of G of size $|M| = \nu(G)$, a contradiction. So, $a'b' \in E(G)$. But now $\big(M \setminus \{aa', bb', cc'\}\big) \cup \{a'b', abc, dc'\}$ is an IPP of G of size $|M| = \nu(G)$, again a contradiction. ♦

It now remains to show that G is chordal. Suppose otherwise, and fix an induced cycle $C = c_0 c_1 \dots c_{k-1} c_0$ (with $k \geq 4$ and with indices taken modulo k) of G.

Claim 2.7.2. *For every maximum matching M of G, either all vertices of C are M-saturated or $E(C) \cap M = \emptyset$.*

Proof of Claim 2.7.2. Suppose otherwise, and fix a maximum matching M of G such that for some $i, j \in \{0, 1, \dots, k-1\}$, c_i is M-unsaturated and $c_j c_{j+1} \in M$. Clearly, $i \notin \{j, j+1\}$. We may assume that M, c_i, and $c_j c_{j+1}$ were chosen so that the path $P := c_i c_{i+1} \dots c_j$ is as short as possible. By symmetry, we may assume that $i = 0$ (and consequently, $j \in \{1, \dots, k-2\}$). Proposition 2.1 now guarantees that c_0 is the unique M-unsaturated vertex of G, and in particular, c_1 is M-saturated. So, there exists some $c_1' \in V(G) \setminus \{c_1\}$ such that $c_1 c_1' \in M$. Since c_0 is M-unsaturated and adjacent to c_1, Proposition 2.1 guarantees that $c_0 c_1' \in E(G)$. Now $\{c_0, c_1, c_1'\}$ induces a triangle in G; since $C = c_0 c_1 \dots c_{k-1} c_0$ is an induced cycle of G of length $k \geq 4$, we see that $c_1' \notin V(C)$, and in particular, $j \neq 1$. But now $M_1 := \big(M \setminus \{c_1 c_1'\}\big) \cup \{c_0 c_1'\}$ is a maximum matching of G, and c_1 is M_1-unsaturated. Meanwhile, we have that $c_jc_{j+1} \in M_1$, and the path $P' := c_1c_2 \dots c_j$ is shorter than the path $P = c_0c_1 \dots c_j$, contrary to the minimality of P. ⧫

Using Proposition 2.3, we fix a perfect matching M of $G \backslash c_0$. By Claim 2.7.2, we have that $E(C) \cap M = \emptyset$. We now deduce that there exist pairwise distinct vertices $c_1', \dots, c_{k-1}' \in V(G) \setminus V(C)$ such that $c_1c_1', \dots, c_{k-1}c_{k-1}' \in M$.

Claim 2.7.3. *$c_0c_1', c_0c_{k-1}' \in E(G)$ and $c_{k-2}c_{k-1}' \notin E(G)$.*

Proof of Claim 2.7.3. Since the M-unsaturated vertex c_0 is adjacent to the endpoint c_1 of the matching edge $c_1c_1' \in M$, Proposition 2.1 guarantees that $c_0c_1' \in E(G)$. Analogously, $c_0c_{k-1}' \in E(G)$. Finally, we have that $c_{k-2}c_{k-1}' \notin E(G)$, for otherwise, vertices $c_0, c_{k-2}, c_{k-1}, c_{k-1}'$ would induce a diamond in G, contrary to the fact that, by Claim 2.7.1, G is diamond-free. ⧫

By Claim 2.7.3, we have that $c_{k-2}c_{k-1}' \notin E(G)$. Now, fix the smallest index $i \in \{1, \dots, k-2\}$ such that $c_ic_{i+1}' \notin E(G)$. By the minimality of i, and by Claim 2.7.3, we have that $c_jc_{j+1}' \in E(G)$ for all $j \in \{0, \dots, i-1\}$. But now

$$\mathcal{I} \quad := \quad \{c_jc_{j+1}' \mid 0 \leq j \leq i-1\} \cup \big(M \setminus \{c_jc_j' \mid 1 \leq j \leq i+1\}\big) \cup \{c_ic_{i+1}c_{i+1}'\}$$

is an IPP of G of size $\nu(G)$, contrary to the fact that $\text{ipp}(G) = \nu(G) + 1$. □

Proposition 2.8. *Let G be a connected graph that has at most one block that is not an odd complete graph. Assume furthermore that if B is a block of G that is not an odd complete graph, then B is even and IPP-extremal. Then G is IPP-extremal.*

Proof. We may assume inductively that for any connected graph G' such that $|V(G')| < |V(G)|$, if the following hold:

- G' has at most one block that is not an odd complete graph, and
- if B' is a block of G' that is not an odd complete graph, then B' is even and IPP-extremal,

then G' is IPP-extremal.

Suppose first that G is biconnected, so that G has exactly one block, namely G itself. If G is even, then the result is immediate. On the other hand, if G is odd, then Proposition 2.7 guarantees that G is an odd complete graph, and we are done.

Suppose now that G is not biconnected. Then G has at least two leaf-blocks, and by hypothesis, at least one of them is an odd complete graph. Let C be the vertex set of this block, and let v be the unique cut-vertex of G that belongs to C. Set $G' := G \setminus \big(C \setminus \{v\}\big)$. Clearly, G' is connected, and by Proposition 2.5(a), the blocks of G are precisely $G[C]$ and the blocks of G'. So, by the induction hypothesis, G' is IPP-extremal. But now Proposition 2.5(b) guarantees that G is IPP-extremal. □

Proposition 2.9. *Let G be a connected graph that has at most one block that is not an odd complete graph. Assume furthermore that if B is a block of G that is not an odd complete graph, then B is even and IndPP-extremal. Then G is IndPP-extremal.*

Proof. The proof is completely analogous to that of Proposition 2.8: we simply replace all instances of "IPP" by "IndPP," and we use Proposition 2.5(c) instead of Proposition 2.5(b).[4] □

Proposition 2.8 yields the backward direction of Theorem 1.6, and Proposition 2.9 yields the backward direction of Theorem 1.7. It remains to prove the forward direction of the two theorems.

Given a graph G and a vertex $v \in V(G)$, we say that an IPP $\mathcal{I}$ of G is *v-extendable* if $|\mathcal{I}| \leq \nu(G)$ and v is an endpoint of the unique path in $\mathcal{I}$ that contains v (possibly, this path is a one-vertex path whose only vertex is v).

Proposition 2.10. *Let G be a biconnected graph, and let $v \in V(G)$. Assume that $G \setminus v$ has a perfect matching, and that $\text{ipp}(G) \leq \nu(G)$. Then G admits a v-extendable IPP.*

---

[4] Note that Proposition 2.7 (used in the proof of Proposition 2.8) does in fact apply to IndPP-extremal graphs, since by Proposition 1.4, IndPP-extremal graphs are IPP-extremal.

Proof. We assume toward a contradiction that G does not admit a v-extendable IPP.

Claim 2.10.1. *The vertex v is not mixed on any edge of any perfect matching of $G \setminus v$.*

Proof of Claim 2.10.1. Fix a perfect matching M of $G \setminus v$, and assume toward a contradiction that v is mixed on some edge $xy \in M$. By symmetry, we may assume that v is adjacent to x and nonadjacent to y. But then $\mathcal{I} := \big(M \setminus \{xy\}\big) \cup \{vxy\}$ is a v-extendable IPP of G, a contradiction. ♦

Claim 2.10.2. *For every edge xx' that belongs to some perfect matching of $G \setminus v$, no vertex in $N_G(v) \setminus \{x, x'\}$ is mixed on xx'.*

Proof of Claim 2.10.2. Fix a perfect matching M of $G \setminus v$, and fix an edge $xx' \in M$. Suppose toward a contradiction that some vertex $u \in N_G(v) \setminus \{x, x'\}$ is mixed on xx'. By symmetry, we may assume that u is adjacent to x and nonadjacent to x'. Since M is a perfect matching of $G \setminus v$, we know that there exists some vertex $u' \in V(G) \setminus \{v, u, x, x'\}$ such that $uu' \in M$. Since $u \in N_G(v)$, Claim 2.10.1 guarantees that $u' \in N_G(v)$. But now $\mathcal{I} := \big(M \setminus \{xx', uu'\}\big) \cup \{uxx', vu'\}$ is a v-extendable IPP of G, a contradiction. ♦

Claim 2.10.3. *For all perfect matchings M of $G \setminus v$, and all distinct edges $xx', yy' \in M$ such that $x, x', y, y' \in N_G(v)$, we have that xx' and yy' are either complete or anticomplete to each other.*

Proof of Claim 2.10.3. Fix a perfect matching M of $G \setminus v$, and fix distinct edges $xx', yy' \in M$ such that $x, x', y, y' \in N_G(v)$. We may assume that xx' is not anticomplete to yy', for otherwise we are done. By symmetry, we may assume that x is adjacent to y. Since $x \in N_G(v)$ and $yy' \in M$, Claim 2.10.2 guarantees that x is complete to yy'. In particular, $y' \in N_G(v)$ is adjacent to x; so, since $xx' \in M$, Claim 2.10.2 guarantees that y' is complete to xx'. But now $x' \in N_G(v)$ is adjacent to y', and so since $yy' \in M$, Claim 2.10.2 guarantees that x' is complete to yy'. We have now shown that xx' is complete to yy', and we are done. ♦

Claim 2.10.4. *$N_G(v)$ can be partitioned into two or more nonempty cliques, pairwise anticomplete to each other.*

Proof of Claim 2.10.4. It suffices to show that v is not simplicial in G, and that $G[N_G(v)]$ contains no induced three-vertex path.[5]

First, suppose toward a contradiction that v is simplicial in G. Fix any minimum IPP $\mathcal{I}$ of G; by hypothesis, we have that $|\mathcal{I}| = \text{ipp}(G) \leq \nu(G)$. Fix the (unique) path $P \in \mathcal{I}$ such that $v \in V(P)$. Clearly, every internal vertex of P has two nonadjacent neighbors in P; since the path P is isometric and therefore induced in G, it follows that no internal vertex of P is simplicial in G. So, v is not an internal vertex of P, and it must therefore be an endpoint of P. But now $\mathcal{I}$ is a v-extendable IPP of G, a contradiction. This proves that v is not simplicial in G.

Now, suppose toward a contradiction that $G[N_G(v)]$ contains an induced three-vertex path xyz. Fix a perfect matching M of $G \setminus v$ (such an M exists by hypothesis). Since $z \in N_G(v)$ is mixed on xy, Claim 2.10.2 guarantees that $xy \notin M$. Similarly, since $x \in N_G(v)$ is mixed on yz, Claim 2.10.2 guarantees that $yz \notin M$. Since M is a perfect matching of $G \setminus v$, it follows that there exist pairwise distinct vertices $x', y', z' \in V(G) \setminus \{v, x, y, z\}$ such that $xx', yy', zz' \in M$. Since $x, y, z \in N_G(v)$, Claim 2.10.1 guarantees that $x', y', z' \in N_G(v)$.

We now have that $x, x', y, y', z, z' \in N_G(v)$, that $xx', yy', zz' \in M$, and that xyz is an induced path in G. So, by Claim 2.10.3, we have that $\{y, y'\}$ is complete to $\{x, x', z, z'\}$, and that $\{x, x'\}$ and $\{z, z'\}$ are anticomplete to each other. In particular, $x'y'z'$ is an induced path in G. But we now see that $\mathcal{I} := \big(M \setminus \{xx', yy', zz'\}\big) \cup \{xyz, x'y'z', v\}$ is a v-extendable IPP of G, a contradiction. This proves that $G[N_G(v)]$ contains no induced three-vertex path, and we are done. ♦

---

[5] Let us explain why this is enough. Suppose that we have proven that v is not simplicial in G (i.e. $N_G(v)$ is not a clique, and in particular, $N_G(v) \neq \emptyset$), and that $G[N_G(v)]$ contains no induced three-vertex path. Let C be a connected component of G, and suppose that C is not a complete graph. Then C contains two nonadjacent vertices, call them c_1 and c_2. Fix an induced path P between c_1 and c_2 in C. This path contains at least three vertices, and so some induced subpath P' of P is a three-vertex path. Clearly, the three-vertex P' is an induced path in $G[N_G(v)]$, a contradiction. Thus, all connected components of $G[N_G(v)]$ are complete, and since v is not simplicial in G, it follows that $G[N_G(v)]$ has at least two connected components. In other words, $N_G(v)$ can be partitioned into two or more nonempty cliques, pairwise anticomplete to each other.

Using Claim 2.10.4, we fix a partition $(C_1, \dots, C_k)$, with $k \geq 2$, of $N_G(v)$ into nonempty cliques, pairwise anticomplete to each other.

Claim 2.10.5. *For every perfect matching M of $G \setminus v$, and every edge $xx' \in M$, either $x, x' \in V(G) \setminus N_G[v]$, or there exists some $i \in \{1, \dots, k\}$ such that $x, x' \in C_i$.*

Proof of Claim 2.10.5. Fix a perfect matching M of $G \setminus v$, and fix an edge $xx' \in M$. By Claim 2.10.1, either $x, x' \in N_G(v)$ or $x, x' \in V(G) \setminus N_G[v]$. We may assume that $x, x' \in N_G(v)$, for otherwise we are done. The result now follows from the fact that $C_1, \dots, C_k$ form a partition of $N_G(v)$ into cliques, pairwise anticomplete to each other. ♦

Claim 2.10.6. *No vertex in $V(G) \setminus N_G[v]$ has more than one neighbor in any one of the cliques $C_1, \dots, C_k$.*

Proof of Claim 2.10.6. Suppose otherwise. By symmetry, we may assume that some vertex $u \in V(G) \setminus N_G[v]$ has at least two neighbors, call them x and y, in C_1.

We first construct a perfect matching M of $G \setminus v$ such that $xy \in M$. Fix any perfect matching M_0 of $G \setminus v$ (such an M_0 exists by hypothesis). If $xy \in M_0$, then we simply set $M := M_0$. So, suppose that $xy \notin M_0$. Then there exist distinct vertices $x', y' \in V(G) \setminus \{v, x, y\}$ such that $xx', yy' \in M_0$. Since $x, y \in C_1$, Claim 2.10.5 guarantees that $x', y' \in C_1$. But C_1 is a clique, and so $x'y' \in E(G)$. We now set $M := \big(M_0 \setminus \{xx', yy'\}\big) \cup \{xy, x'y'\}$, and we observe that M is a perfect matching of $G \setminus v$ such that $xy \in M$.

Now, since M is a perfect matching of $G \setminus v$, there exists some $u' \in V(G) \setminus \{v, u, x, y\}$ such that $uu' \in M$. Since $u \in V(G) \setminus N_G[v]$, Claim 2.10.1 guarantees that $u' \in V(G) \setminus N_G[v]$. Since $x, y \in N_G(v)$ are both adjacent to u, and since $uu' \in M$, Claim 2.10.2 guarantees that $\{x, y\}$ is complete to $\{u, u'\}$. But now $\mathcal{I} := \big(M \setminus \{xy, uu'\}\big) \cup \{vxu, yu'\}$ is a v-extendable IPP of G, a contradiction. ♦

Now, since G is biconnected, we see that $G \setminus v$ is connected. So, there exists an induced path in $Q := q_0 q_1 \dots q_t q_{t+1}$ ($t \geq 1$) such that $q_0 \in C_1$ and $q_{t+1} \in C_2 \cup \dots \cup C_k$, whereas $q_1, \dots, q_t$ (the internal vertices of Q) belong to $V(G) \setminus N_G[v]$. By symmetry, we may assume that $q_{t+1} \in C_2$. Further, fix a perfect matching M of $G \setminus v$ (such an M exists by hypothesis), and we fix vertices $q_0', q_1', \dots, q_t', q_{t+1}' \in V(G) \setminus \{v\}$ such that $q_0 q_0', q_1 q_1', \dots, q_t q_t', q_{t+1} q_{t+1}' \in M$.

Claim 2.10.7. *All the following hold:*

(1) vertices $q_0, q_1, \dots, q_t, q_{t+1}, q_0', q_1', \dots, q_t', q_{t+1}'$ are pairwise distinct (and in particular, no edge of Q belongs to M);

(2) for all $i \in \{0, \dots, t-1\}$, we have that $q_i q_{i+1}' \in E(G)$;

(3) $q_t q_{t+1}' \notin E(G)$.

Proof of Claim 2.10.7. It is clear that $q_0, q_1, \dots, q_t, q_{t+1}$ are pairwise distinct, and it is also clear that $q_0', q_1', \dots, q_t', q_{t+1}'$ are pairwise distinct. Thus, to prove (1), it is enough to show that $q_0', q_1', \dots, q_t', q_{t+1}' \notin V(Q)$.

First, since $q_0 \in C_1$, $q_{t+1} \in C_2$, and $q_0 q_0', q_{t+1} q_{t+1}' \in M$, Claim 2.10.5 guarantees that $q_0' \in C_1$ and $q_{t+1}' \in C_2$. In particular, $q_0', q_{t+1}' \notin V(Q)$. Furthermore, since $q_1, \dots, q_t \in V(G) \setminus N_G[v]$, Claim 2.10.1 guarantees that $q_1', \dots, q_t' \in V(G) \setminus N_G[v]$. Further, since $q_{t+1}, q_{t+1}' \in C_2$, and since $q_t \in V(G) \setminus N_G[v]$ is adjacent to q_{t+1}, Claim 2.10.6 guarantees that q_t is nonadjacent to q_{t+1}'; this proves (3). It remains to prove (1) and (2).

Since $q_0 \in N_G(v)$ is adjacent to q_1, and since $q_1 q_1' \in M$, Claim 2.10.2 guarantees that $q_0 q_1' \in E(G)$. Consequently, $q_1' \notin V(Q)$ (because the path Q is induced). We have now shown that $q_0', q_1' \notin V(Q)$ and $q_0 q_1' \in E(G)$. Fix the largest index $s \in \{0, \dots, t-1\}$ such that $q_0', \dots, q_{s+1}' \notin V(Q)$, and such that for all indices $i \in \{0, \dots, s\}$, we have that $q_i q_{i+1}' \in E(G)$. We have already shown that $q_{t+1}' \notin V(Q)$, and so if $s = t-1$, then (1) and (2) hold, and we are done. We may therefore assume that $s \leq t-2$.

Since $q_{s+1}' \notin V(Q)$, we know that $q_{s+2}' \neq q_{s+1}$.[6] But now $q_{s+1} q_{s+2}' \in E(G)$, for otherwise,

$$\mathcal{I} \ := \ \big(M \setminus \{q_0 q_0', q_1 q_1', \dots, q_{s+2} q_{s+2}'\}\big) \cup \{v q_0', q_0 q_1', \dots, q_s q_{s+1}'\} \cup \{q_{s+1} q_{s+2} q_{s+2}'\}$$

---

[6] Indeed, edges $q_{s+1} q_{s+1}'$ and $q_{s+2} q_{s+2}'$ both belong to the matching M, and so either these two edges are equal, or they have no endpoints in common. So, if we had $q_{s+2}' = q_{s+1}$, then it would follow that $q_{s+2} = q_{s+1}'$, contrary to the fact that $q_{s+2} \in V(Q)$ and $q_{s+1}' \notin V(Q)$.

would be a v-extendable IPP of G, a contradiction. Since the path Q is induced in G, this further implies that $q'_{s+2} \notin V(Q)$. We have now derived a contradiction to the maximality of the index s. ♦

Using Claim 2.10.7, we now see that

$$\mathcal{I} \quad := \quad \big(M \setminus \{q_0q'_0, q_1q'_1, \dots, q_tq'_t, q_{t+1}q'_{t+1}\}\big) \cup \{vq'_0, q_0q'_1, \dots, q_{t-1}q'_t, q_tq_{t+1}q'_{t+1}\}$$

is a v-extendable IPP of G, a contradiction. This completes the argument. □

Proposition 2.11. *Let G be a connected IPP-extremal graph. Then G has at most one block that is not an odd complete graph. Moreover, if G has exactly one block that is not an odd complete graph, then that block is even.*

Proof. We may assume inductively that the proposition is true for graphs on fewer than $|V(G)|$ many vertices. More precisely, we assume inductively that for all connected IPP-extremal graphs G' such that $|V(G')| < |V(G)|$, the graph G' has at most one block that is not an odd complete graph, and moreover, if G' has exactly one block that is not an odd complete graph, then that block is even.

Suppose first that G is biconnected, so that G has exactly one block, namely G itself. If G is even, then the result is immediate, whereas if G is odd, then Proposition 2.7 guarantees that G is an odd complete graph, and we are done.

From now on, we assume that G is not biconnected. Therefore, G has at least two leaf-blocks.

Claim 2.11.1. *G has at most one even leaf-block.*

Proof of Claim 2.11.1. Suppose otherwise, and fix distinct even blocks B_1 and B_2 of G. Further, for each index $i \in \{1, 2\}$, let v_i be the (unique) cut-vertex of G that belongs to B_i (it is possible that $v_1 = v_2$).

Suppose first that G is odd. Then Proposition 2.3 guarantees that $G \setminus v_1$ admits a perfect matching. But this is impossible because $G \setminus v_1$ has at least one odd connected component, namely $B_1 \setminus v_1$.

Thus, G is even. Then Proposition 2.2 guarantees that G admits a perfect matching, call it M. For each $i \in \{1, 2\}$, we fix $b_i \in V(G) \setminus \{v_i\}$ such that $b_iv_i \in M$, and we observe that $b_i \in V(B_i) \setminus \{v_i\}$, for otherwise, $B_i \setminus v_i$ would be an odd connected component of $G \setminus \{b_i, v_i\}$, contrary to the fact that $G \setminus \{b_i, v_i\}$ has a perfect matching (namely $M \setminus \{b_iv_i\}$). In particular, matching edges b_1v_1 and b_2v_2 are distinct (and consequently have no common endpoints). Now, let P be a shortest path between b_1 and b_2 in G. Clearly, the path P contains the matching edges b_1v_1 and b_2v_2. Set $M' := \{uv \in M \mid u, v \notin V(P)\}$, and note that $M' \subseteq M \setminus \{b_1v_1, b_2v_2\}$. But now

$$\mathcal{I} \quad := \quad \{P\} \cup M' \cup \big\{v \in V(G) \setminus V(P) \mid \exists u \in V(P) \text{ s.t. } uv \in M\big\}$$

is an IPP of G, and consequently, $\text{ipp}(G) \leq |\mathcal{I}| \leq |M| - 1 = \nu(G) - 1$, contrary to Proposition 2.2. ♦

Since G has at least two leaf-blocks, and since at most one of them is even (by Claim 2.11.1), we see that G has an odd leaf-block, call it B. Let v be the (unique) cut-vertex of G that belongs to B. Now, if G is even, then set $U := \emptyset$; and if G is odd, then let U be a singleton whose only member is some vertex of $V(G) \setminus V(B)$. Propositions 2.2 and 2.3 now guarantee that $G_U := G \setminus U$ has a perfect matching, call it M, and that $M \cup U$ is a minimum IPP of G, i.e. $\text{ipp}(G) = |M| + |U|$. By construction, we have that $v \notin U$, and so v is M-saturated; fix $v^* \in V(G) \setminus \{v\}$ such that $vv^* \in M$. Set $M_B := M \cap E(B)$ and $M'_B := M \setminus E(B)$.

Claim 2.11.2. *We have that $v^* \notin V(B)$ and $vv^* \in M'_B$. Moreover, M_B is a perfect matching of $B \setminus v$, and consequently, $\nu(B) = |M_B|$.*

Proof of Claim 2.11.2. If we had that $v^* \in V(B)$, then $B \setminus \{v, v^*\}$ would be an odd connected component of $G_U \setminus \{v, v^*\}$, contrary to the fact that $G_U \setminus \{v, v^*\}$ has a perfect matching, namely $M \setminus \{vv^*\}$. Thus, $v^* \notin V(B)$. It follows that, $vv^* \notin M_B$, and consequently, $vv^* \in M'_B$. But now $B \setminus v$ is a connected component of $G_U \setminus \{v, v^*\}$, and $M \setminus \{vv^*\}$ is a perfect matching of G_U. So, $\big(M \setminus \{vv^*\}\big) \cap E(B)$ is a perfect matching of $B \setminus v$, and clearly, $\big(M \setminus \{vv^*\}\big) \cap E(B) = \big(M \cap E(B)\big) \setminus \{vv^*\} = M_B \setminus \{vv^*\} = M_B$. ♦

Claim 2.11.3. *B is IPP-extremal.*

Proof of Claim 2.11.3. Suppose otherwise, so that (by Proposition 1.3) we have that $\mathrm{ipp}(B) \leq |V(B)| - \nu(B) - 1$. Since $B \setminus v$ has a perfect matching (by Claim 2.11.2), we see that $|V(B)| = 2\nu(B) + 1$, and we deduce that $\mathrm{ipp}(B) \leq \nu(B)$. Hence, by Proposition 2.10, B admits a v-extendable IPP $\mathcal{I}_B$. By the definition of a v-extendable IPP, and by Claim 2.11.2, we have that $|\mathcal{I}_B| \leq \nu(B) = |M_B|$. Further, fix $P \in \mathcal{I}_B$ such that $v \in V(P)$. Since $\mathcal{I}_B$ is v-extendable, we know that v is an endpoint of P; set $P = vp_1 \dots p_t$ $(t \geq 0)$. By Claim 2.11.2, we have that $v^* \notin V(B)$ and $vv^* \in M'_B$, and we deduce that

$$\mathcal{I} \quad := \quad \{v^*vp_1 \dots p_t\} \cup \big(\mathcal{I}_B \setminus \{P\}\big) \cup \big(M'_B \setminus \{vv^*\}\big) \cup U$$

is an IPP of G. But then

$$\begin{aligned} \mathrm{ipp}(B) \quad \leq \quad |\mathcal{I}| \quad &= \quad 1 + (|\mathcal{I}_B| - 1) + (|M'_B| - 1) + |U| \\ &= \quad |\mathcal{I}_B| + |M'_B| + |U| - 1 \\ &\leq \quad |M_B| + |M'_B| + |U| - 1 \\ &= \quad |M_B \cup M'_B \cup U| - 1 \\ &= \quad |M \cup U| - 1, \end{aligned}$$

contrary to the fact that $M \cup U$ is a minimum IPP of G. ♦

Since the block B is odd (and also biconnected, by the definition of a block), Claim 2.11.3 and Proposition 2.7 together imply that B is an odd complete graph. Set $G' := G \setminus \big(V(B) \setminus \{v\}\big)$. By Proposition 2.5, G' is IPP-extremal and the blocks of G are precisely the block B and the blocks of G'. Since B is an odd complete graph, the result now follows immediately from the induction hypothesis applied to G'. ☐

Proposition 2.12. *Let G be a connected graph. Then both the following hold:*

(a) if G is IPP-extremal, then so are all its blocks;

(b) if G is IndPP-extremal, then so are all its blocks.

Proof. We may assume inductively that the proposition is true for graphs on fewer than $|V(G)|$ vertices. More precisely, we assume inductively that for all connected graphs G' such that $|V(G')| < |V(G)|$, if G' is IPP-extremal (resp. IndPP-extremal), then all blocks of G' are IPP-extremal (resp. IndPP-extremal). We may further assume G is not biconnected, for otherwise, G has exactly one block (namely itself), and the result is immediate. Therefore, G has at least two leaf-blocks.

Suppose first that no leaf-block of G is an odd complete graph. Since G has at least two leaf-blocks, Proposition 2.11 now implies that G is not IPP-extremal. Consequently, by Proposition 1.4, G is not IndPP-extremal either. Therefore, both (a) and (b) are vacuously true, and we are done.

From now on, we assume that G has a leaf-block, call it B, that is an odd complete graph. Set $G' := G \setminus \big(V(B) \setminus \{v\}\big)$. Clearly, G' is connected, and furthermore, Proposition 2.5 guarantees that the blocks of G are precisely the block B and the blocks of G'. Since B is complete, it is clear that B is both IPP-extremal and IndPP-extremal . On the other hand, Proposition 2.5 guarantees that if G is IPP-extremal (resp. IndPP-extremal), then G' is also IPP-extremal (resp. IndPP-extremal). The result now follows immediately from the induction hypothesis applied to G'. ☐

We are now ready to prove Theorems 1.6 and 1.7. As we have previously pointed out, Proposition 2.8 yields the backward direction of Theorem 1.6, and Proposition 2.9 yields the backward direction of Theorem 1.7. On the other hand, the forward direction of both theorems follows from Propositions 2.11 and 2.12.[7]

---

[7] In the case of Theorem 1.7, we are also using the fact that IndPP-extremal graphs are IPP-extremal (by Proposition 1.4), which means that Proposition 2.11 also applies to IndPP-extremal graphs.

3 Extremal graphs

In this section, we use Theorem 1.6 (resp. Theorem 1.7) to obtain a complete characterization of connected odd graphs that are IPP-extremal (resp. IndPP-extremal), as well as a characterization of IPP-extremal block graphs (resp. IndPP-extremal block graphs). We begin with a simple proposition.

Proposition 3.1. *If a connected graph G has exactly k even blocks, then $|V(G)|$ and $k+1$ have the same parity. In particular, any connected graph with no even blocks is odd, and any connected graph with exactly one even block is even.*

Proof. Fix a connected graph G with exactly k even blocks, and assume inductively that all connected graphs G' such that $|V(G')| < |V(G)|$, if G' has exactly k' even blocks, then $|V(G')|$ and $k'+1$ have the same parity. We must show that $|V(G)|$ and $k+1$ have the same parity.

If G is biconnected, then G has exactly one block, namely G itself, and the result is immediate. So, we may assume that G is not biconnected. Then G has at least two leaf-blocks. Let B be some leaf-block of G. Clearly, $|V(B)| \geq 2$. Let v be the (unique) cut-vertex of G that belongs to B, and set $G' := G \setminus \big(V(B) \setminus \{v\}\big)$. Then the blocks of G are precisely the blocks of G', plus the block B, and obviously, G' is connected. Moreover, $|V(G)| = |V(G')|+|V(B)|-1$, and in particular, $|V(G')| < |V(G)|$. Let k' be the number of even blocks of G'. By the induction hypothesis, $|V(G')|$ and $k'+1$ have the same parity.

Suppose first that the leaf-block B is even. Then the number of even blocks of G is $k = k'+1$, whereas $|V(G)|$ and $|V(G')|$ have the opposite parity (because $|V(G)| = |V(G')|+|V(B)|-1$, and $|V(B)|$ is even). Since $|V(G')|$ and $k'+1$ have the same parity, it follows that $|V(G)|$ and $k+1$ also have the same parity.

Suppose now that the leaf-block B is odd. Then the number of even blocks of G is $k = k'$, whereas $|V(G)|$ and $|V(G')|$ have the same parity (because $|V(G)| = |V(G')| + |V(B)| - 1$, and $|V(B)|$ is odd). Since $|V(G')|$ and $k'+1$ have the same parity, so do $|V(G)|$ and $k+1$. ☐

Theorems 1.6 and 1.7, together with Proposition 3.1, readily yield the following four corollaries.

Corollary 3.2. *Let G be a connected odd graph. Then the following are equivalent:*

- *G is IPP-extremal;*
- *G is IndPP-extremal;*
- *every block of G is an odd complete graph (and in particular, G is a block graph).*

Corollary 3.3. *Let G be a connected even graph. Then the following are equivalent:*

- *G is IPP-extremal;*
- *G has exactly one block that is not an odd complete graph, and that block is even and IPP-extremal.*

Corollary 3.4. *Let G be a connected even graph. Then the following are equivalent:*

- *G is IndPP-extremal;*
- *G has exactly one block that is not an odd complete graph, and that block is even and IndPP-extremal.*

Corollary 3.5. *Let G be a connected block graph. Then the following are equivalent:*

- *G is IPP-extremal;*
- *G is IndPP-extremal;*
- *at most one block of G is even.*

4 Concluding remarks and open questions

We have introduced an upper bound on the isometric path partition number and induced path partition number of a graph in terms of the matching number: any graph G satisfies $\mathrm{indpp}(G) \leq \mathrm{ipp}(G) \leq |V(G)| - \nu(G)$ (see Proposition 1.3). By Proposition 1.5, a graph is IPP-extremal (resp. IndPP-extremal) if and only if all its components are IPP-extremal(resp. IndPP-extremal). Theorem 1.6 and Theorem 1.7 provide a characterization of connected IPP-extremal graphs and IndPP-extremal graphs, respectively, in terms of their blocks. Corollary 3.2 gives a complete characterization of all connected odd graphs that are IPP-extremal and that are IndPP-extremal: a connected odd graph G is IPP-extremal (resp. G is IndPP-extremal) if and only if it is a block graph containing only odd blocks. On the other hand, by Corollary 3.3 (resp. Corollary 3.4), a connected even graph G is IPP-extremal (resp. IndPP-extremal) if and only if it contains exactly one block that is not a complete odd graph, and this one block is an even IPP-extremal graph (resp. IndPP-extremal graph). This reduces the problem of fully characterizing the IPP-extremal graphs and IndPP-extremal graphs to the even biconnected case. Thus, we propose the following question.

Question 4.1. *Which biconnected even graphs are IPP-extremal and are IndPP-extremal?*

We note that some obvious examples of biconnected even graphs that are IPP-extremal and IndPP-extremal include even complete graphs, the diamond, and the cycle C_4.

Our proof of Theorems 1.6 and 1.7 made heavy use of inductive arguments. However, similar inductive arguments are unlikely to yield an answer to Question 4.1, and this is essentially because the IPP and IndPP numbers are not monotone with respect to the induced subgraph relation. More precisely, for an induced subgraph H of a graph G, any one of the following is possible: $\mathrm{ipp}(H) < \mathrm{ipp}(G)$ (resp. $\mathrm{indpp}(H) < \mathrm{indpp}(G)$), $\mathrm{ipp}(H) = \mathrm{ipp}(G)$ (resp. $\mathrm{indpp}(H) = \mathrm{indpp}(G)$), $\mathrm{ipp}(H) > \mathrm{ipp}(G)$ (resp. $\mathrm{indpp}(H) > \mathrm{indpp}(G)$). It is easy to come up with examples of the first two possibilities (just consider complete graphs). For a concrete example of the third possibility, consider the graph G and its induced subgraph H represented in Figure 2. It is easy to see that $\mathrm{indpp}(G) = \mathrm{ipp}(G) = 2$, whereas $\mathrm{ipp}(H) = 3$, $\mathrm{indpp}(H) = \mathrm{ipp}(H) = 3$. For the latter, we simply observe that no IPP (resp. IndPP) of H can contain more than two leaves of H; since H contains five leaves, it follows that $\mathrm{ipp}(H) \geq 3$ (resp. $\mathrm{indpp}(H) \geq 3$). On the other hand, an IPP of H, which is also an IndPP of H, of cardinality 3 is shown in Figure 2.

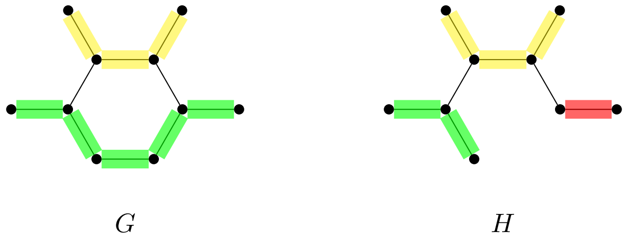

Figure 2: An illustration of a graph G and its subgraph H where $\mathrm{ipp}(G) < \mathrm{ipp}(H)$ and $\mathrm{indpp}(G) < \mathrm{indpp}(H)$.

Other related parameters in this line of study include the (unrestricted) path partition (denoted as pp); which has been studied in [21]. Notice that every isometric path is an induced path, and every induced path is a path; this, together with Proposition 1.3 implies that, for any graph G, $\mathrm{pp}(G) \leq \mathrm{indpp}(G) \leq \mathrm{ipp}(G) \leq |V(G)| - \nu(G)$. One can observe that every IndPP-extremal graph with respect to this upper bound is also IPP-extremal, but the converse is not true. A simple example for this case is a wheel graph on 6-vertices (see Figure 1). On the other hand, the cycle C_4 is both IPP-extremal and IndPP-extremal, but not PP-extremal with respect to our upper bound. Thus, we propose the following question.

Question 4.2. *Find a characterization of graphs that are PP-extremal.*

Our upper bound for the IPP number raises an interesting algorithmic question as follows.

Question 4.3. *Given a graph G and an integer k, does there exist an IPP of G of cardinality of at most $|V(G)| - \nu(G) - k$?*

Questions of this nature have been the subject of growing interest in the algorithmic community in recent years. Such formulations often form the foundation for parameterized complexity studies and are valuable in exploring the tractability and kernelizability of graph problems under certain structural constraints.

Acknowledgment

We thank Prof. Florent Foucaud for the initial discussions on this problem. The original version of this manuscript dealt only with IPP. We thank an anonymous referee for pointing out that most of our results about IPP generalize to IndPP, an observation that has been incorporated into this paper. We thank both referees for their careful reading of our manuscript and helpful suggestions.

References

[1] H.-J. Bandelt and H. M. Mulder. Distance-hereditary graphs. *Journal of Combinatorial Theory, Series B*, 41(2):182–208, 1986.

[2] C. Berge. Path partitions in directed graphs. In *North-Holland Mathematics Studies*, volume 75, pages 59–63. Elsevier, 1983.

[3] F. T. Boesch and J. F. Gimpel. Covering points of a digraph with point-disjoint paths and its application to code optimization. *Journal of the ACM (JACM)*, 24(2):192–198, 1977.

[4] D. Chakraborty, H. Muller, S. Ordyniak, F. Panolan, and M. Rychlicki. Covering and partitioning of split, chain and cographs with isometric paths. *Leibniz International Proceedings in Informatics. 49th International Symposium on Mathematical Foundations of Computer Science*, 2024.

[5] G. Chartrand, J. McCanna, N. Sherwani, M. Hossain, and J. Hashmi. The induced path number of bipartite graphs. *Ars combinatoria*, 37:191–208, 1994.

[6] D. G. Corneil, B. Dalton, and M. Habib. Ldfs-based certifying algorithm for the minimum path cover problem on cocomparability graphs. *SIAM Journal on Computing*, 42(3):792–807, 2013.

[7] H. Fernau, F. Foucaud, K. Mann, U. Padariya, and R. R. KN. Parameterizing path partitions. *Theoretical Computer Science*, 1028:115029, 2025.

[8] S. L. Fitzpatrick. Aspects of domination and dynamic domination. 1997.

[9] S. L. Fitzpatrick, R. J. Nowakowski, D. Holton, and I. Caines. Covering hypercubes by isometric paths. *Discrete Mathematics*, 240(1):253–260, 2001.

[10] T. Gallai and A. N. Milgram. Verallgemeinerung eines graphentheoretischen satzes von rédei. *Acta Sc. Math*, 21:181–186, 1960.

[11] M. R. Garey, D. S. Johnson, and R. E. Tarjan. The planar hamiltonian circuit problem is NP-complete. *SIAM Journal on Computing*, 5(4):704–714, 1976.

[12] L. Guo, Y. Deng, K. Liao, Q. He, T. Sellis, and Z. Hu. A fast algorithm for optimally finding partially disjoint shortest paths. In *Twenty-Seventh International Joint Conference on Artificial Intelligence (IJCAI-18), Stockholm, Sweden, July 13-19, 2018*, pages 1456–1462. International Joint Conferences on Artificial Intelligence, 2018.

[13] K.-i. Kawarabayashi, Y. Kobayashi, and B. Reed. The disjoint paths problem in quadratic time. *Journal of Combinatorial Theory, Series B*, 102(2):424–435, 2012.

[14] M. Lafond and V. Moulton. Path partitions of phylogenetic networks. *Theoretical Computer Science*, 1024:114907, 2025.

[15] H.-O. Le, V. B. Le, and H. Müller. Splitting a graph into disjoint induced paths or cycles. *Discrete applied mathematics*, 131(1):199–212, 2003.

[16] P. Manuel. Revisiting path-type covering and partitioning problems. *arXiv preprint arXiv:1807.10613*, 2018.

[17] P. D. Manuel. On the isometric path partition problem. *Discussiones Mathematicae: Graph Theory*, 41(4):1077–1089, 2021.

[18] S. C. Ntafos and S. L. Hakimi. On path cover problems in digraphs and applications to program testing. *IEEE Transactions on Software Engineering*, (5):520–529, 1979.

[19] J.-J. Pan and G. J. Chang. Induced-path partition on graphs with special blocks. *Theoretical Computer Science*, 370(1):121–130, 2007.

[20] N. Robertson and P. D. Seymour. Graph minors. xiii. the disjoint paths problem. *Journal of combinatorial theory, Series B*, 63(1):65–110, 1995.

[21] Z. Skupién. Path partitions of vertices and hamiltonicity of graphs. In *Proceedings of the Second Czechoslovakian Symposium on Graph Theory, Prague*, 1974.