Perfect graphs with no balanced skew-partition are 2-clique-colorable

Irena Penev *

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Abstract

A graph G is *perfect* if for all induced subgraphs H of G, $\chi(H) = \omega(H)$. A graph G is *Berge* if neither G nor its complement contains an induced odd cycle of length at least five. The Strong Perfect Graph Theorem [9] states that a graph is perfect if and only if it is Berge. The Strong Perfect Graph Theorem was obtained as a consequence of a decomposition theorem for Berge graphs [4, 9], and one of the decompositions in this decomposition theorem was the "balanced skew-partition." A *clique-coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no inclusion-wise maximal clique of G of size at least two is monochromatic, and the *clique-chromatic number* of G, denoted by $\chi_C(G)$, is the smallest number of colors needed to clique-color G. There exist graphs of arbitrarily large clique-chromatic number, but it is not known whether the clique-chromatic number of perfect graphs is bounded. In this paper, we prove that every perfect graph that does not admit a balanced skew-partition is 2-clique colorable. The main tool used in the proof is a decomposition theorem for "tame Berge trigraphs" due to Chudnovsky et al. [11]

1 Introduction

All graphs in this paper are finite and simple. Given a graph G, we denote by $V(G)$ the vertex-set of G, by $E(G)$ the edge-set of G, and by $\chi(G)$ the chromatic number of G. A *clique* of a graph G is a set of pairwise adjacent vertices of G. The *clique number* of G, denoted by $\omega(G)$, is the size of a maximum clique of G. An assignment of colors to the vertices of G is said to be a *clique-coloring* of G provided that no inclusion-wise maximal clique of G of size at least two is colored monochromatically. (As usual,

*Université de Lyon, LIP, UMR 5668, ENS de Lyon - CNRS - UCBL - INRIA. E-mail: irena.penev@ens-lyon.fr. This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR). Partially supported by ANR project Stint under reference ANR-13-BS02-0007.

a set $S \subseteq V(G)$ is said to be *monochromatic* with respect to a given coloring of G provided that no two vertices of S receive different colors.) A *k-clique-coloring* of G is a clique-coloring $q : V(G) \to \{1, \dots, k\}$ of G, and G is said to be *k-clique-colorable* if it admits a k-clique-coloring. The *clique-chromatic number* of G, denoted by $\chi_C(G)$, is the smallest number k such that G is k-clique-colorable. It is clear that any proper coloring of a graph G is a clique-coloring of G, and so $\chi_C(G) \leq \chi(G)$. It is also clear that if G is a triangle-free graph (that is, if $\omega(G) \leq 2$), then any clique-coloring of G is also a proper coloring of G, and consequently, $\chi_C(G) = \chi(G)$. (Since there exist triangle-free graphs of arbitrarily large chromatic number [20, 25], this implies that there exist triangle-free graphs of arbitrarily large clique-chromatic number.) However, if G contains a triangle, then $\chi(G)$ may be much larger than $\chi_C(G)$. For instance, if $n \geq 2$, then $\chi(K_n) = n$, while $\chi_C(K_n) = 2$. Furthermore, there exist connected graphs of arbitrarily large clique number and clique-chromatic number; indeed, Bacsó et al. [2] showed that the line graphs of complete graphs can have an arbitrarily large clique-chromatic number.

A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. A graph is *imperfect* if it is not perfect. Given graphs G and H, we say that G is *H-free* provided that G does not contain (an isomorphic copy of) H as an induced subgraph. If $\mathcal{H}$ is a family of graphs, a graph G is said to be *$\mathcal{H}$-free* provided that for all graphs $H \in \mathcal{H}$, G is H-free. A *hole* in a graph G is an induced cycle of G of length at least four; a hole of G is *odd* if its length is odd. A graph G is said to be *Berge* if neither G nor $\overline{G}$ (the complement of G) contains an odd hole. The Strong Perfect Graph Theorem [9] states that a graph is perfect if and only if it is Berge. As pointed out above, there exist graphs of arbitrarily large clique-chromatic number. However, it remains an open problem to determine whether the clique-chromatic number of perfect graphs is bounded (this question was raised, for example, in [14]). Progress in this direction has been made in some special cases. For example, Bacsó et al. [2] proved that all {claw, odd-hole}-free graphs are 2-clique-colorable, and Défossez [12] proved that all {bull, odd-hole}-free graphs are 2-clique-colorable. (The *claw* is the complete bipartite graph $K_{1,3}$, and the *bull* is the five-vertex graph consisting of a triangle and two vertex-disjoint pendant edges.) In addition, Bacsó et al. [2] proved that "generalized split graphs" are 3-clique-colorable (a *generalized split graph* is a graph G whose vertex-set can be partitioned into sets $A, B_1, \dots, B_k$ such that in either G or $\overline{G}$, $A, B_1, \dots, B_k$ are all cliques and there are no edges between any two of $B_1, \dots, B_k$; it is easy to see that generalized split graphs are perfect). Using a result of Prömel and Steger [21] that "almost all" C_5-free graphs are generalized split graphs, Bacsó et al. [2] deduced that "almost all" perfect graphs are 3-clique-colorable. On the other hand, there exist perfect graphs whose clique-chromatic number is three: one well-known example is the graph (which we will call $C_9^{\triangle}$) with vertex-set $\{a_1, a_2, \dots, a_9\}$ and edge-set $\{a_1a_2, a_2a_3, a_3a_4, \dots, a_8a_9, a_9a_1\} \cup \{a_3a_6, a_6a_9, a_9a_3\}$. (Thus, the graph $C_9^{\triangle}$ is obtained from the cycle of length nine by choosing three evenly spaced vertices and adding edges between them.) All this suggests that it might be true that all perfect graphs are 3-clique-colorable.

One direction of the Strong Perfect Graph Theorem [9] cited above is a simple exercise: it is easy to see that odd cycles of length at least five and their complements are imperfect, and consequently, all perfect graphs are Berge. The proof of the other direction ("all Berge graphs are perfect") is over a hundred pages long, and its main ingredient is a decomposition theorem for Berge graphs. One version of this decomposition theorem was proven in [9], and a stronger version was proven in [4]. The stronger version of this decomposition theorem is stated below (we remark that we have not yet defined all the terms that appear in the statement of this theorem).

Theorem 1.1. *[4] Let G be a Berge graph. Then at least one of the following holds:*

- *G or $\overline{G}$ is a bipartite graph;*
- *G or $\overline{G}$ is the line graph of a bipartite graph;*
- *G is a double-split graph;*
- *G or $\overline{G}$ admits a proper 2-join;*
- *G admits a balanced skew-partition.*

Let us now define the balanced skew-partition. (We will define other terms used in Theorem 1.1 in a slightly more general context in section 3.) Given a graph G and a set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. The *length* of a path is the number of edges that it contains, and a path is *odd* if its length is odd. A *skew-partition* of G is a partition (X, Y) of $V(G)$ such that $G[X]$ and $\overline{G}[Y]$ are both disconnected. A skew-partition (X, Y) of G is *balanced* if it satisfies the following two conditions:

- G contains no induced odd path of length greater than one whose endpoints belong to Y and all of whose interior vertices belong to X;
- $\overline{G}$ contains no induced odd path of length greater than one whose endpoints belong to X and all of whose interior vertices belong to Y.

Note that (X, Y) is a skew-partition (respectively: balanced skew-partition) of a graph G if and only if (Y, X) is a skew-partition (respectively: balanced skew-partition) of $\overline{G}$. Thus, G admits a balanced skew-partition if and only if $\overline{G}$ does. The main result of this paper is stated below (its proof is given in section 6).

Theorem 1.2. *Every perfect graph that does not admit a balanced skew-partition is* 2*-clique-colorable.*

The balanced skew-partition decomposition is notoriously difficult to work with, and it presents the main obstacle to using Theorem 1.1 for (for instance) constructing combinatorial polynomial-time algorithms for perfect (equivalently: Berge) graphs. The difficulty of dealing with balanced skew-partitions has led to the study of perfect graphs that do not admit this particular decomposition. Trotignon [23] proved a decomposition theorem for Berge graphs (related to, but different from, the decomposition theorems from [4, 9]) and used it to construct a polynomial-time algorithm that decides whether a Berge graph admits a balanced skew-partition. (See [24] for some further algorithmic consequences of the decomposition theorem from [23]. We also remark that Berge graphs can be recognized in polynomial time [8]; the algorithm from [8] does not rely on decomposition theorems from [4, 9, 23].) Chudnovsky et al. [11] proved a decomposition theorem for "tame Berge trigraphs," a theorem that is particularly well suited for the study of Berge (tri)graphs that do not admit a balanced skew-partition (a more detailed discussion of this can be found in section 3 of the present paper). Formal definitions are given in sections 2 and 3, but informally, a trigraph is a generalization of a graph in which a pair of distinct vertices can be "strongly adjacent," "strongly anti-adjacent," or "semi-adjacent." One natural way to think of this is to consider strongly adjacent pairs and strongly anti-adjacent pairs to be edges and non-edges, respectively, whereas semi-adjacent pairs have undetermined adjacency. (In particular, a graph can be seen as a trigraph that has no semi-adjacent pairs.) Trigraphs were originally introduced by Chudnovsky [4, 5] as a tool for studying decompositions of Berge graphs; in fact Theorem 1.1 cited above is a consequence of a theorem about Berge trigraphs proven in [4]. (Later, trigraphs were also used to study the structure of claw-free graphs [10] and bull-free graphs [6, 7].) The use of trigraphs in [11] lead to a theorem that is simpler than the one from [23]. Furthermore, Chudnovsky et al. [11] used the decomposition theorem from that same paper to give a combinatorial polynomial-time coloring algorithm for perfect graphs that do not admit a balanced skew-partition. (We remark that a polynomial-time coloring algorithm for perfect graphs was given in [17]. However, the algorithm from [17] relies on the ellipsoid method and is therefore not combinatorial.) The decomposition theorem from [11] was also used by Lagoutte and Trunck [19] to study clique-stable set separation in perfect graphs that do not admit a balanced skew-partition. In the present paper, we use the decomposition theorem from [11] to prove Theorem 1.2.

We complete the Introduction by giving an outline of the paper. In section 2, we define "trigraphs" and "trigraph clique-coloring," and we prove a few preliminary results about the latter. In section 3, we state Theorem 3.1, which is the analogue of Theorem 1.2 for trigraphs (in fact, Theorem 1.2 is an immediate corollary of Theorem 3.1). In section 3, we also state those results of [11] that we need for the purposes of proving Theorem 3.1, and (using the results of [4]) we slightly strengthen the main structural result of [11] (see Theorem 3.10). We remark that the definitions of trigraphs and related concepts given in sections 2 and 3 were mostly taken from [11] (with some minor modifications). However, to the author's knowledge, trigraph clique-coloring has not previously been studied (in particular, the definition of trigraph clique-coloring given in section 2 of the present paper is new). In section 4, we prove that (with the exception of two small trigraphs) the "basic" trigraphs from Theorem 3.10 are 2-clique-colorable, and in section 5, we deal with decompositions that appear in Theorem 3.10. In section 6, we use Theorem 3.10 and the results of sections 4 and 5 to prove Theorem 3.1, from which we then easily derive Theorem 1.2 (the main result of this paper). Finally, in section 7, we discuss a few open problems.

2 Trigraphs and clique-coloring

Given a set S and a non-negative integer k, we denote by $\binom{S}{k}$ the set of all subsets of S of size k. A *trigraph* is an ordered pair $G = (V(G), \theta_G)$, where $V(G)$ is a finite set, called the *vertex-set* of G, and $\theta_G : \binom{V(G)}{2} \rightarrow \{-1, 0, 1\}$ is a function, called the *adjacency function* of G. The *null trigraph* is the trigraph whose vertex-set is empty, and a *non-null* trigraph is any trigraph whose vertex-set is non-empty. If G is a trigraph, then members of $V(G)$ are called *vertices* of G, and if $u, v \in V(G)$ are distinct, then:

- if $\theta_G(\{u, v\}) = 1$, we say that uv is a *strongly adjacent pair* of G, or that u and v are *strongly adjacent* in G, or that u is *strongly adjacent* to v in G, or that v is a *strong neighbor* of u in G, or that u and v are the *endpoints of a strongly adjacent pair* of G;

- if $\theta_G(\{u, v\}) = 0$, we say that uv is a *semi-adjacent pair* of G, or that u and v are *semi-adjacent* in G, or that u is *semi-adjacent* to v in G, or that v is a *weak neighbor* of u in G, or that u and v are the *endpoints of a semi-adjacent pair* of G;

- if $\theta_G(\{u, v\}) = -1$, we say that uv is a *strongly anti-adjacent pair* of G, or that u and v are *strongly anti-adjacent* in G, or that u is *strongly anti-adjacent* to v in G, or that v is a *strong anti-neighbor* of u in G, or that u and v are the *endpoints of a strongly anti-adjacent pair* of G;

- if $\theta_G(\{u, v\}) \geq 0$, we say that uv is an *adjacent pair* of G, or that u and v are *adjacent* in G, or that u is *adjacent* to v in G, or that v is a *neighbor* of u in G, or that u and v are the *endpoints of an adjacent pair* of G;

- if $\theta_G(\{u,v\}) \leq 0$, we say that uv is an *anti-adjacent pair* of G, or that u and v are *anti-adjacent* in G, or that u is *anti-adjacent* to v in G, or that v is an *anti-neighbor* of u in G, or that u and v are the *endpoints of an anti-adjacent pair* of G.

We remark that in some papers on trigraphs (for instance, in [11]), semi-adjacent pairs are called "switchable pairs." If G is a trigraph and $A, B \subseteq V(G)$ are disjoint sets, a *semi-adjacent pair between A and B* is a semi-adjacent pair of G whose one endpoint belongs to A and whose other endpoint belongs to B. Note that a semi-adjacent pair is simultaneously an adjacent pair and an anti-adjacent pair. Note also that any graph can be thought of as a trigraph: a graph is simply a trigraph with no semi-adjacent pairs. Indeed, if G is a graph with vertex-set $V(G)$ and edge-set $E(G)$, we can turn G into a trigraph by defining the adjacency function $\theta_G : V(G) \to \{-1,0,1\}$ of G by setting

$$\theta_G(\{u,v\}) = \begin{cases} 1 & \text{if} \quad uv \in E(G) \\ \\ -1 & \text{if} \quad uv \notin E(G) \end{cases}$$

for all distinct $u, v \in V(G)$.

Given a trigraph G, a vertex $u \in V(G)$, and a set $X \subseteq V(G) \smallsetminus \{u\}$, we say that u is *complete* (respectively: *strongly complete, anti-complete, strongly anti-complete*) to X in G provided that u is adjacent (respectively: strongly adjacent, anti-adjacent, strongly anti-adjacent) to every vertex of X in G. Given a trigraph G and disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (respectively: *strongly complete, anti-complete, strongly anti-complete)* to Y in G provided that every vertex of X is complete (respectively: strongly complete, anti-complete, strongly anti-complete) to Y in G.

Isomorphism between trigraphs is defined in the natural way. The *complement* of a trigraph $G = (V(G), \theta_G)$ is the trigraph $\overline{G} = (V(\overline{G}), \theta_{\overline{G}})$ such that $V(\overline{G}) = V(G)$ and $\theta_{\overline{G}} = -\theta_G$. Thus, $\overline{G}$ is obtained from G by turning all strongly adjacent pairs of G into strongly anti-adjacent pairs, and turning all strongly anti-adjacent pairs of G into strongly adjacent pairs; semi-adjacent pairs of G remain semi-adjacent in $\overline{G}$.

Given trigraphs G and $\widetilde{G}$, we say that $\widetilde{G}$ is a *semi-realization* of G provided that $V(\widetilde{G}) = V(G)$, and that for all distinct $u, v \in V(\widetilde{G}) = V(G)$, we have that if $\theta_G(\{u,v\}) = 1$ then $\theta_{\widetilde{G}}(\{u,v\}) = 1$, and if $\theta_G(\{u,v\}) = -1$ then $\theta_{\widetilde{G}}(\{u,v\}) = -1$. Thus, a semi-realization of a trigraph G is any trigraph that can be obtained from G by “deciding” the adjacency of some semi-adjacent pairs of G, that is, by possibly turning some semi-adjacent pairs of G into strongly adjacent or strongly anti-adjacent pairs. (In particular, every trigraph is a semi-realization of itself.) A *realization* of a trigraph G is a graph that is a semi-realization of G. Thus, a realization of a trigraph G is any graph that can be obtained by “deciding” the adjacency of all semi-adjacent pairs of G, that is, by turning each semi-adjacent pair of G into an edge or a non-edge. Clearly, if a trigraph G has m semi-adjacent pairs, then G has exactly 2^m realizations. The *full realization* of a trigraph G is the graph obtained from G by turning all semi-adjacent pairs of G into strongly adjacent pairs (i.e. edges), and the *null realization* of G is the graph obtained from G by turning all semi-adjacent pairs of G into strongly anti-adjacent pairs (i.e. non-edges). Note that the complement of the full realization of a trigraph G is equal to the null realization of the complement of G. Similarly, the complement of the null realization of a trigraph G is equal to the full realization of the complement of G.

Given a trigraph G and a set $X \subseteq V(G)$, the *subtrigraph of G induced by X*, denoted by $G[X]$, is the trigraph with vertex-set X and adjacency function $\theta_G \upharpoonright \binom{X}{2}$. If $v_1, \dots, v_k$ are vertices of a trigraph G, we often write $G[v_1, \dots, v_k]$ instead of $G[\{v_1, \dots, v_k\}]$.

A *light vertex* of a trigraph G is a vertex $w \in V(G)$ that satisfies all the following:

- w has exactly two weak neighbors in G, call them u and v;
- w is strongly anti-complete to $V(G) \setminus \{u, v, w\}$ in G;
- uv is a strongly anti-adjacent pair in G;
- neither u nor v has a weak neighbor in $V(G) \setminus \{u, v, w\}$ in G.

A *heavy vertex* of a trigraph G is a vertex $w \in V(G)$ that satisfies all the following:

- w has exactly two weak neighbors in G, call them u and v;
- w is strongly complete to $V(G) \setminus \{u, v, w\}$ in G;
- uv is a strongly adjacent pair in G;
- neither u nor v has a weak neighbor in $V(G) \setminus \{u, v, w\}$ in G.

Note that w is a light vertex of a trigraph G if and only if w is a heavy vertex of $\overline{G}$. A trigraph G is said to be *tame* provided that no vertex of G has more than two weak neighbors in G, and that every vertex of G that has exactly two weak neighbors is either light or heavy. A trigraph G is said to be *monogamous* provided that no vertex of G has more than one weak neighbor in G. Thus, every graph is a monogamous trigraph, and every monogamous trigraph is a tame trigraph. Furthermore, note that a trigraph is tame (respectively: monogamous) if and only if its complement is tame (respectively: monogamous). It is also clear that if G is a tame (respectively: monogamous) trigraph, then all semi-realizations of G and all induced subtrigraphs of G are tame (respectively: monogamous) trigraphs. Most of this paper will be devoted to tame trigraphs; in particular, the decomposition theorem from [11] (which we state in section 3) deals with tame "Berge" trigraphs ("Berge" trigraphs are defined in section 3).

A *clique* (respectively: *strong clique*, *stable set*, *strongly stable set*) of a trigraph G is a set of pairwise adjacent (respectively: strongly adjacent, anti-adjacent, strongly anti-adjacent) vertices of G. Note that any subset of $V(G)$ of size at most one is both a strong clique and a strongly stable set of G. Note also that if $S \subseteq V(G)$, then S is a (strong) clique of G if and only if S is a (strongly) stable set of $\overline{G}$. Note furthermore that if K is a strong clique and S is a stable set of G, then $|K \cap S| \leq 1$; similarly, if K is a clique and S is a strongly stable set of G, then $|K \cap S| \leq 1$. However, if K is a clique and S is a stable set of G, then we are only guaranteed that vertices in $K \cap S$ are pairwise semi-adjacent to each other, and it is possible that $|K \cap S| \geq 2$. A *(strong) triangle* is a (strong) clique of size three. A clique K in a trigraph G is said to be *important* provided that $|K| \geq 2$ and that no vertex in $V(G) \setminus K$ is strongly complete to K in G. Note that if G is a trigraph, then a set $K \subseteq V(G)$ is an important clique of G if and only if there exists a realization $\widetilde{G}$ of G such that K is an inclusion-wise maximal clique of size at least two of $\widetilde{G}$. (In fact, more is true: if G is a trigraph, then a set $K \subseteq V(G)$ is an important clique of G if and only if K is an inclusion-wise maximal clique of size at least two of the realization of G obtained by turning all semi-adjacent pairs of G, both of whose endpoints lie in K, into edges, and turning all other semi-adjacent pairs of G into non-edges.) Note also that if $\widetilde{G}$ is a semi-realization of a trigraph G, then every important clique of $\widetilde{G}$ is an important clique of G, but an important clique of G need not be an important clique of $\widetilde{G}$ (indeed, an important clique K of G need not be a clique of $\widetilde{G}$, and even if K is a clique of $\widetilde{G}$, it may fail to be important in $\widetilde{G}$). Furthermore, it is clear that if G is a graph, then a set $K \subseteq V(G)$ is an important clique of G if and only if K is an inclusion-wise maximal clique of G of size at least two.

A *clique-coloring* of a trigraph G is an assignment of colors to the vertices of G in such a way that no important clique of G is colored monochromatically. A *k-clique-coloring* of a trigraph G is a clique-coloring $q : V(G) \to \{1, \dots, k\}$ of G. A trigraph G is *k-clique-colorable* if there exists a k-clique-coloring of G, and the *clique-chromatic number* of G, denoted by $\chi_C(G)$, is the smallest number k such that G is k-clique-colorable. We now prove three easy propositions concerning clique-coloring of trigraphs. The first of these propositions (namely, Proposition 2.1) will be used extensively throughout the paper. The remaining two propositions (Propositions 2.2 and 2.3) will not be used in subsequent sections, but they may be useful for gaining a better understanding of trigraph clique-coloring, and we include them for this reason.

Proposition 2.1. *Let k be a positive integer, and let G be a trigraph. Then G is k-clique-colorable if and only if there exists a partition $(S_1, \dots, S_k)$ of $V(G)$ into k (possibly empty) sets such that no important clique of G is included in any one of $S_1, \dots, S_k$.*

Proof. This follows immediately from the definition of a k-clique-coloring. (The sets $S_1, \dots, S_k$ are the color classes of a k-clique-coloring of G.) □

Proposition 2.2. *Let G be a trigraph, and let $\widetilde{G}$ be a semi-realization of G. Then every clique-coloring of G is a clique-coloring of $\widetilde{G}$ as well, and consequently, $\chi_C(\widetilde{G}) \leq \chi_C(G)$.*

Proof. The first statement follows from the fact that every important clique of $\widetilde{G}$ is also an important clique of G. The second statement follows from the first. □

Proposition 2.3. *Let G be a trigraph, and let q be an assignment of colors to the vertices of G. Then q is a clique-coloring of G if and only if q is a clique-coloring of all the realizations of G.*

Proof. The "only if" part follows from Proposition 2.2. For the "if" part, suppose that q is a clique-coloring of all the realizations of G. To show that q is a clique-coloring of G, fix a clique K of G of size at least two, monochromatic with respect to q. Let G_K be the realization of G that turns all semi-adjacent pairs of G, both of whose endpoints lie in K, into edges, and that turns all other semi-adjacent pairs of G into non-edges. Then K is a clique of G_K of size at least two, monochromatic with respect to q. By assumption, q is a clique-coloring of G_K, and consequently, K is not a maximal clique of G_K. Fix $v \in V(G) \smallsetminus K$ such that $K \cup \{v\}$ is a clique of G_K. By the construction of G_K then, v is strongly complete to K in G, and so K is not an important clique of G. This proves that q is a clique-coloring of G. □

Proposition 2.2 implies that for every trigraph G, we have that

$$\max\{\chi_C(\widetilde{G}) \mid \widetilde{G} \text{ is a realization of } G\} \quad \leq \quad \chi_C(G). \tag{1}$$

One might wonder whether the inequality from (1) can be turned into an equality. The answer to this question is "no," even if one restricts one's attention to tame trigraphs. To see this, consider the trigraph E_1 with vertex-set $\{a, b, c\}$ in which ab is a strongly adjacent pair, and ac, bc are semi-adjacent pairs. (The trigraph E_1 appears in the statement of Theorem 3.1, our main result for trigraphs. We remark that c is a heavy vertex of E_1.) It is easy to see that $\chi_C(E_1) = 3$, even though the clique-chromatic number of every realization of E_1 is two. We remark that we have not been able to determine whether this sort of "anomaly" can occur in the context of monogamous trigraphs, that is, we have not been able to determine whether for all monogamous trigraphs G, one has $\chi_C(G) = \max\{\chi_C(\widetilde{G}) \mid \widetilde{G} \text{ is a realization of } G\}$. We do, however, know that there are monogamous trigraphs whose clique-chromatic number is greater than both the clique-chromatic number of their full realization, and the clique-chromatic number of their null realization. One example is the trigraph G with vertex-set $\{a_1, a_2, a_3, a_4, a_5\}$ in which $a_1a_2, a_2a_3, a_3a_4, a_4a_5$ are strongly adjacent pairs, a_1a_5, a_2a_4 are semi-adjacent pairs, and $a_1a_3, a_1a_4, a_2a_5, a_3a_5$ are strongly anti-adjacent pairs. Then $\chi_C(G) = 3$, even though the clique-chromatic number of both the full realization and the null realization of G is two. However, one realization of G is a chordless cycle of length five, and the clique-chromatic number of this realization of G is three.

3 Theorems 3.1 and 3.10, and results of [11]

In this section, we state Theorem 3.1, the main result of this paper for trigraphs. (Theorem 3.1 is proven in section 6. Theorem 1.2, stated in the Introduction, is a special case of Theorem 3.1.) In this section, we also state (and slightly strengthen) certain results from [11] that we need in this paper. We begin with some definitions.

A trigraph is said to be a *path* provided that its vertex-set can be ordered as $\{p_0, \dots, p_k\}$ (where $k \geq 0$), so that for all distinct $i, j \in \{0, \dots, k\}$, if $|i - j| = 1$ then p_ip_j is an adjacent pair, and if $|i - j| > 1$ then p_ip_j is an anti-adjacent pair; such a path is denoted by $p_0 - \dots - p_k$. The *endpoints* of a path $p_0 - \dots - p_k$ are the vertices p_0 and p_k (we also say that the path $p_0 - \dots - p_k$ is *between* p_0 and p_k), and the interior vertices of $p_0 - \dots - p_k$ are the vertices $p_1, \dots, p_{k-1}$. The *length* of a path $p_0 - \dots - p_k$ is k. (Thus, a path of length k contains $k + 1$ vertices.) A path is said to be *odd* if its length is odd. A *path* in a trigraph G is an induced subtrigraph P of G such that P is a path. Given disjoint sets $A, B \subseteq V(G)$, a path $p_0 - \dots - p_k$ of G is said to be *between* A and B provided that $p_0 \in A$, $p_k \in B$, and $p_1, \dots, p_{k-1} \notin A \cup B$. A trigraph G is said to be *connected* provided that for all distinct $u, v \in V(G)$, there is a path in G between u and v. A *component* of a non-null trigraph G is a maximal connected induced subtrigraph of G. Note that a trigraph is connected if and only if its full realization is connected, and that an induced subtrigraph C of a trigraph G is a component of G if and only if the full realization of C is a component of the full realization of G.

An induced subtrigraph H of a trigraph G is a *hole* in G provided that the vertex-set of H can be ordered as $\{h_1, \dots, h_k\}$ (with $k \geq 4$), so that for all distinct $i, j \in \{1, \dots, k\}$, if $|i-j| = 1$ or $|i-j| = k-1$ then $h_i h_j$ is an adjacent pair, and if $1 < |i-j| < k-1$ then $h_i h_j$ is an anti-adjacent pair; such a hole is often denoted by $h_1 - h_2 - \dots - h_k - h_1$. The *length* of a hole is the number of vertices that it contains; a hole is said to be *odd* if its length is odd. A trigraph G is said to be *Berge* if neither G nor $\overline{G}$ contains an odd hole. By definition, a trigraph is Berge if and only if its complement is Berge. It is also clear that if a trigraph is Berge, then so are all its semi-realizations and all its induced subtrigraphs. Furthermore, note that a trigraph is Berge if and only if all its realizations are Berge. (We remark that the class $\mathcal{F}$ from [11] is precisely the class of tame Berge trigraphs. In the present paper, however, rather than saying "G belongs to the class $\mathcal{F}$," we simply say "G is a tame Berge trigraph.")

A *skew-partition* of a trigraph G is a partition (X, Y) of $V(G)$ such that neither $G[X]$ nor $\overline{G}[Y]$ is connected. A skew-partition (X, Y) of a trigraph G is said to be *balanced* provided that both the following hold:

- G contains no odd path of length greater than one whose endpoints belong to Y and all of whose interior vertices belong to X;
- $\overline{G}$ contains no odd path of length greater than one whose endpoints belong to X and all of whose interior vertices belong to Y.

Note that if (X, Y) is a (balanced) skew-partition of a trigraph G, then (Y, X) is a (balanced) skew-partition of $\overline{G}$. Note also that if (X, Y) is a (balanced) skew-partition of a trigraph G, then (X, Y) is (balanced) skew-partition of all semi-realizations of G. We say that a trigraph G *admits a (balanced) skew-partition* if there exists a (balanced) skew-partition (X, Y) of G.

We remind the reader that E_1 is the trigraph with vertex-set $\{a, b, c\}$ in which ab is a strongly adjacent pair, and ac, bc are semi-adjacent pairs. Let E_2 be the trigraph with vertex-set $\{a, b, c, d\}$ in which ab, cd are strongly adjacent pairs, ac, bc are semi-adjacent pairs, and ad, bd are strongly anti-adjacent pairs. It is easy to see that E_1 and E_2 are both tame Berge trigraphs that do not admit a balanced skew-partition. Using the fact that, in both E_1 and E_2, the sets $\{a, b\}$, $\{b, c\}$, and $\{c, a\}$ are important cliques (and therefore cannot be colored monochromatically by any clique-coloring of E_1 or E_2), we deduce that $\chi_C(E_1) = \chi_C(E_2) = 3$. Our main theorem for trigraphs (stated below, and proven in section 6) states that, with the exception of E_1 and E_2, all tame Berge trigraphs that do not admit a balanced skew-partition are 2-clique-colorable.

Theorem 3.1. *Every tame Berge trigraph that does not admit a balanced skew-partition, and that is isomorphic to neither E_1 nor E_2, is 2-clique-colorable.*

A class of trigraphs is *hereditary* if it is closed under isomorphism and induced subtrigraphs. (In particular then, a class of graphs is hereditary if it is closed under isomorphism and induced subgraphs.) It may be worth pointing out that the class of tame Berge trigraphs that do not admit a balanced skew-partition (that is, the class with which Theorem 3.1 is concerned) is not hereditary. In fact, even the class of Berge (equivalently: perfect) graphs that do not admit a balanced skew-partition is not hereditary. To see this, note that graphs that are cycles of even length are Berge and do not admit a balanced skew-partition, whereas graphs that are paths of length at least three do admit a balanced skew-partition. Do, however, note that the class of tame Berge trigraphs that do not admit a balanced skew-partition is closed under complementation, that is, if G is a tame Berge trigraph that does not admit a balanced skew-partition, then $\overline{G}$ is also a tame Berge trigraph that does not admit a balanced skew-partition.

As stated in the Introduction, the main tool for proving Theorem 3.1 is the decomposition theorem from [11] for tame Berge trigraphs, and its slight strengthening (namely, Theorem 3.10), which we prove in this section. Informally, the decomposition theorem from [11] states that if G is a tame Berge trigraph, then either G is "basic," or one of $G, \overline{G}$ admits a "decomposition." One of the decompositions is the balanced skew-partition. The other is the "proper 2-join" decomposition, which we now define.

A *2-join* of a trigraph G is a partition (X_1, X_2) of $V(G)$ such that there exist pairwise disjoint sets $A_1, B_1, C_1, A_2, B_2, C_2 \subseteq V(G)$ that satisfy all the following:

- for each $i \in \{1, 2\}$, $|X_i| \geq 3$ and $X_i = A_i \cup B_i \cup C_i$;
- the sets A_1, B_1, A_2, B_2 are all non-empty;
- A_1 is strongly complete to A_2 and strongly anti-complete to B_2 in G;
- B_1 is strongly complete to B_2 and strongly anti-complete to A_2 in G;
- C_1 is strongly anti-complete to $A_2 \cup B_2 \cup C_2$ in G;
- C_2 is strongly anti-complete to $A_1 \cup B_1 \cup C_1$ in G;

- for each $i \in \{1, 2\}$, if $|A_i| = |B_i| = 1$, then the full realization of $G[X_i]$ is not a chordless path of length two between the unique vertex of A_i and the unique vertex of B_i.

Under these circumstances, we say that $(A_1, B_1, C_1, A_2, B_2, C_2)$ is a *split* of the 2-join (X_1, X_2) of G. The 2-join (X_1, X_2) of G is *proper* if for each $i \in \{1, 2\}$, every component of $G[X_i]$ meets both A_i and B_i. The 2-join (X_1, X_2) is said to be *odd* (respectively: *even*) if for each $i \in \{1, 2\}$, every path in $G[X_i]$ between A_i and B_i is of odd (respectively: even) length. The following was proven in [11] (see 2.4 of [11]).

Proposition 3.2. *[11] Let G be a Berge trigraph. Then every proper 2-join of G is either odd or even.*

We remark that 3.1 of [11] implies that every 2-join of a tame Berge trigraph that does not admit a balanced skew-partition is proper (and therefore, by Proposition 3.2, either odd or even). The following two propositions, proven in [11] (see 3.1, 4.1, and 4.2 from [11]) will be of use to us for "decomposing" tame Berge trigraphs that do not admit a balanced skew-partition, but do admit a 2-join.

Proposition 3.3. *[11] Let G be a trigraph that admits an odd 2-join (X_1, X_2) with split $(A_1, B_1, C_1, A_2, B_2, C_2)$. Let a_1, b_1, a_2, b_2 be pairwise distinct vertices that do not belong to $V(G)$, and for each $i \in \{1, 2\}$, let G_i be the trigraph on the vertex-set $A_i \cup B_i \cup C_i \cup \{a_{3-i}, b_{3-i}\}$, with adjacency as follows:*

- *$G_i[A_i \cup B_i \cup C_i] = G[A_i \cup B_i \cup C_i]$;*
- *$a_{3-i}b_{3-i}$ is a semi-adjacent pair;*
- *a_{3-i} is strongly complete to A_i and strongly anti-complete to $B_i \cup C_i$;*
- *b_{3-i} is strongly complete to B_i and strongly anti-complete to $A_i \cup C_i$.*

If G is a tame Berge trigraph that does not admit a balanced skew-partition, then the following hold:

- *(X_1, X_2) is a proper 2-join of G;*
- *G_1 and G_2 are tame Berge trigraphs that do not admit a balanced skew-partition;*
- *$|X_i| \geq 4$ for each $i \in \{1, 2\}$.*

Proposition 3.4. *[11] Let G be a trigraph that admits an even 2-join (X_1, X_2) with split $(A_1, B_1, C_1, A_2, B_2, C_2)$. Let $a_1, b_1, c_1, a_2, b_2, c_2$ be pairwise distinct vertices that do not belong to $V(G)$, and for each $i \in \{1, 2\}$, let G_i be the trigraph on the vertex-set $A_i \cup B_i \cup C_i \cup \{a_{3-i}, b_{3-i}, c_{3-i}\}$, with adjacency as follows:*

- *$G_i[A_i \cup B_i \cup C_i] = G[A_i \cup B_i \cup C_i]$;*
- *$a_{3-i}c_{3-i}$ and $b_{3-i}c_{3-i}$ are semi-adjacent pairs;*
- *$a_{3-i}b_{3-i}$ is a strongly anti-adjacent pair;*
- *a_{3-i} is strongly complete to A_i and strongly anti-complete to $B_i \cup C_i$;*
- *b_{3-i} is strongly complete to B_i and strongly anti-complete to $A_i \cup C_i$.*
- *c_{3-i} is strongly anti-complete to $A_i \cup B_i \cup C_i$.*

If G is a tame Berge trigraph that does not admit a balanced skew-partition, then the following hold:

- *(X_1, X_2) is a proper 2-join of G;*
- *G_1 and G_2 are tame Berge trigraphs that do not admit a balanced skew-partition;*
- *$|X_i| \geq 4$ for each $i \in \{1, 2\}$.*

We remark that in Propositions 3.3 and 3.4, the fact that $|X_1|, |X_2| \geq 4$ implies that $6 \leq |V(G_1)|, |V(G_2)| < |V(G)|$. The fact that trigraphs G_1 and G_2 have fewer vertices than G is essential for proofs by induction on the number of vertices.

We next define "basic" trigraphs. A trigraph is *bipartite* if its vertex-set can be partitioned into two (possibly empty) strongly stable sets. A trigraph is *complement-bipartite* if its vertex set can be partitioned into two (possibly empty) strong cliques. Clearly, a trigraph is bipartite if and only if its complement is complement-bipartite. Note that every realization of a bipartite trigraph is a bipartite graph, and that every realization of a complement-bipartite trigraph is a complement-bipartite graph (i.e. a graph whose complement is bipartite). It is easy to verify that bipartite and complement-bipartite trigraphs are Berge.

The *line graph* of a graph H, denoted by $L(H)$, is the graph whose vertex-set is $E(H)$, and such that for all distinct $e_1, e_2 \in E(H)$, e_1 and e_2 are adjacent in $L(H)$ if and only if e_1 and e_2 share an endpoint in H. We say that a trigraph G is a *line trigraph* provided that the full realization of G is (isomorphic to) the line graph of some bipartite graph, and that every clique of G of size at least three is strong. The following two propositions are 2.1 from [11] and 2.1 from [4].

Proposition 3.5. *[11] Let G be a line trigraph. Then every semi-realization of G is a line trigraph, and in particular, every realization of G is the line graph of a bipartite graph.*

Proposition 3.6. *[4] Every line trigraph is Berge.*

Next, a trigraph G is said to be *double-split* provided that its vertex-set can be ordered as $V(G) = \{x_1, x'_1, x_2, x'_2, \dots, x_s, x'_s\} \cup \{y_1, y'_1, y_2, y'_2, \dots, y_t, y'_t\}$ (where $s, t \geq 2$), so that adjacency in G is as follows:

- $\{x_1, x'_1\}, \{x_2, x'_2\}, \dots, \{x_s, x'_s\}$ are cliques in G, pairwise strongly anticomplete to each other;
- $\{y_1, y'_1\}, \{y_2, y'_2\}, \dots, \{y_t, y'_t\}$ are stable sets in G, pairwise strongly complete to each other;
- for all $i \in \{1, 2, \dots, s\}$ and $j \in \{1, 2, \dots, t\}$, one of the following holds:
  - $x_i y_j, x'_i y'_j$ are strongly adjacent pairs, and $x_i y'_j, x'_i y_j$ are strongly anti-adjacent pairs,
  - $x_i y'_j, x'_i y_j$ are strongly adjacent pairs, and $x_i y_j, x'_i y'_j$ are strongly anti-adjacent pairs.

Note that under these circumstances, $\{x_1, x'_1\}, \dots, \{x_s, x'_s\}$ are the vertex-sets of the components of $G[x_1, x'_1, \dots, x_s, x'_s]$, and $\{y_1, y'_1\}, \dots, \{y_t, y'_t\}$ are the vertex-sets of the components of $\overline{G}[y_1, y'_1, \dots, y_t, y'_t]$. Further, it is clear that $(\{x_1, x'_1, \dots, x_s, x'_s\}, \{y_1, y'_1, \dots, y_t, y'_t\})$ is a skew-partition of G; however, since one of $y_1 - x_1 - x'_1 - y'_1$ and $y_1 - x'_1 - x_1 - y'_1$ is an odd path of G, this skew-partition of G is not balanced. Clearly, double-split trigraphs are monogamous, and furthermore, a trigraph is double-split if and only if its complement is double-split. By 2.2 from [4], every double-split trigraph is Berge. Further, it was shown in [23] (see Lemma 4.5 of [23]) that double-split graphs (i.e. double-split trigraphs that have no semi-adjacent pairs) do not admit a balanced skew-partition. In fact, it is not difficult to show that double-split trigraphs do not admit a balanced skew-partition, but we do not use this fact in this paper, and so we omit the proof. A trigraph G is said to be *doubled* if it is an induced subtrigraph of some double-split trigraph. Clearly, doubled trigraphs are monogamous, and furthermore, a trigraph is doubled if and only if its complement is doubled.

We now state the decomposition theorem for tame Berge trigraphs from [11].

Theorem 3.7. *[11] Let G be a tame Berge trigraph. Then at least one of the following holds:*

- *G or $\overline{G}$ is bipartite;*
- *G or $\overline{G}$ is a line trigraph;*
- *G is doubled trigraph;*
- *G admits a balanced skew-partition;*

- G *or* $\overline{G}$ *admits a proper* 2*-join.*

Unfortunately, Theorem 3.7 is not sufficient for the purposes of proving Theorem 3.1. The reason for this is that the graph $C_9^{\triangle}$ (defined in the Introduction) is doubled, and yet $\chi_C(G_9^{\triangle}) = 3$. However, $C_9^{\triangle}$ admits a balanced skew-partition. In view of this, our goal is to strengthen Theorem 3.7 by replacing the word "doubled" with the word "double-split" in the statement of that theorem. We first need a definition. A *star-cutset* of a trigraph G is a set $Y \subseteq V(G)$ such that $(V(G) \smallsetminus Y, Y)$ is a skew-partition of G, and some component of $\overline{G}[Y]$ contains only one vertex. The following is 5.9 from [4].

Lemma 3.8. *[4] If* G *is a Berge trigraph such that* G *or* $\overline{G}$ *admits a star-cutset, then* G *admits a balanced skew-partition.*

Proposition 3.9. *Let* G *be a double-split trigraph, and let* $S \subseteq V(G)$*. Then at least one of the following holds:*

- $G[S]$ *is a double-split trigraph;*
- $G[S]$ *is a bipartite or complement-bipartite trigraph;*
- $G[S]$ *or* $\overline{G}[S]$ *admits a star-cutset.*

Proof. Set $V(G) = \{x_1, x'_1, x_2, x'_2, \dots, x_s, x'_s\} \cup \{y_1, y'_1, y_2, y'_2, \dots, y_t, y'_t\}$ (with $s, t \geq 2$), as in the definition of a double-split trigraph. If S meets at most one component of $G[x_1, x'_1, x_2, x'_2, \dots, x_s, x'_s]$, then it is easy to see that $G[S]$ is complement-bipartite. Similarly, if S meets at most one component of $\overline{G}[y_1, y'_1, y_2, y'_2, \dots, y_t, y'_t]$, then $G[S]$ is bipartite. Thus, we may assume that S meets at least two components of $G[x_1, x'_1, x_2, x'_2, \dots, x_s, x'_s]$, and at least two components of $\overline{G}[y_1, y'_1, y_2, y'_2, \dots, y_t, y'_t]$. If for some $j \in \{1, 2, \dots, t\}$, S contains exactly one of y_j and y'_j, then $S \cap \{y_1, y'_1, y_2, y'_2, \dots, y_t, y'_t\}$ is a star-cutset of $G[S]$, and if for some $i \in \{1, 2, \dots, s\}$, S contains exactly one of x_i and x'_i, then $S \cap \{x_1, x'_1, x_2, x'_2, \dots, x_s, x'_s\}$ is a star-cutset of $\overline{G}[S]$. So we may assume that for all $i \in \{1, 2, \dots, s\}$, either $x_i, x'_i \in S$ or $x_i, x'_i \notin S$, and that for all $j \in \{1, 2, \dots, t\}$, either $y_j, y'_j \in S$ or $y_j, y'_j \notin S$. Since S meets at least two components of $G[x_1, x'_1, x_2, x'_2, \dots, x_s, x'_s]$, and at least two components of $\overline{G}[y_1, y'_1, y_2, y'_2, \dots, y_t, y'_t]$, this implies that $G[S]$ is a double-split trigraph. □

We are now ready to prove a slightly stronger version of Theorem 3.7.

Theorem 3.10. *Let* G *be a tame Berge trigraph. Then at least one of the following holds:*

- G *or* $\overline{G}$ *is bipartite;*
- G *or* $\overline{G}$ *is a line trigraph;*

- *G is a double-split trigraph;*
- *G admits a balanced skew-partition;*
- *G or $\overline{G}$ admits a proper 2-join.*

Proof. This follows from Theorem 3.7, Lemma 3.8, and Proposition 3.9. □

We remark here that by replacing the word "tame" with the word "monogamous" in Theorem 3.10, one obtains 3.2 from [4]. (Note that Theorem 1.1, stated in the Introduction, is an immediate corollary of 3.2 from [4].) One may wonder why we use Theorem 3.10, rather than 3.2 from [4]. The reason for this is that if G is a monogamous Berge trigraph that does not admit a balanced skew-partition, but does admit an even 2-join, there does not appear to be a way to conveniently "decompose" G into two smaller monogamous Berge trigraphs that do not admit a balanced skew-partition. On the other hand, Proposition 3.4 gives us a way to do precisely this in the context of tame trigraphs. This is the reason why we work with tame (rather than monogamous) trigraphs, and why we need Theorem 3.10. We also remark that since all doubled trigraphs are monogamous, Theorem 3.10 could also be obtained as a corollary of 3.2 from [4] and Theorem 3.7. The proof given here is more direct, and we include it for this reason.

We complete this section by proving Theorem 3.1 for the special case of trigraphs that contain at least one heavy vertex.

Lemma 3.11. *Let G be a tame Berge trigraph that does not admit a balanced skew-partition, and that contains at least one heavy vertex. Then either G is isomorphic to one of E_1 and E_2, or G is 2-clique-colorable.*

Proof. Let w be a heavy vertex of G, let u and v be the two weak neighbors of G, and let $X = V(G) \smallsetminus \{u, v, w\}$. Then w is strongly complete to X, uv is a strongly adjacent pair, and there are no semi-adjacent pairs between $\{u, v\}$ and X. If u and v have a common neighbor $x^+ \in X$, then x^+ is strongly complete to $\{u, v, w\}$ and w is strongly complete to X in G, and consequently, no important clique of G is included in either $\{u, v, w\}$ or X; Proposition 2.1 now implies that G is 2-clique-colorable. Next, suppose that u and v have a common anti-neighbor $x^- \in X$. If $X = \{x^-\}$, then G is isomorphic to E_2, and if $\{x^-\} \subsetneq X$, then $\{w\} \cup (X \smallsetminus \{x^-\})$ is a star-cutset of G, and Lemma 3.8 implies that G admits a balanced skew-partition, which is a contradiction. So from now on, we assume that u and v have no common neighbors and no common anti-neighbors in X. Let X_u be the set of all neighbors of u in X, let X_v be the set of all neighbors of v in X. Then $X = X_u \cup X_v$ and $X_u \cap X_v = \emptyset$. We note that X_u is strongly complete to X_v in G, for otherwise, we fix anti-adjacent vertices $u' \in X_u$ and $v' \in X_v$, and we observe that $w - u' - u - v - v' - w$ is an odd hole in G, contrary to the fact that G is Berge. We may assume that at least one of X_u and X_v is non-empty, for otherwise, G is isomorphic to E_1, and we are done. By symmetry, we may assume that $X_u \neq \emptyset$. Then $X_u \neq \emptyset$ is strongly complete to $\{u, w\} \cup X_v$, and u is strongly complete to $\{v\} \cup X_u$, and so no important clique of G is included in either $\{u, w\} \cup X_v$ or $\{v\} \cup X_u$. Proposition 2.1 applied to the partition $(\{u, w\} \cup X_v, \{v\} \cup X_u)$ of $V(G)$ now implies that G is 2-clique-colorable. □

4 Basic trigraphs

In this section, we deal with the basic building blocks of Theorem 3.10, that is, with bipartite and complement-bipartite trigraphs, line trigraphs and their complements, and double-split trigraphs. We show that, with some restrictions, trigraphs from these classes are 2-clique-colorable. Lemma 4.9, proven at the end of this section, essentially constitutes the basis case of the proof by induction of Theorem 3.1.

Proposition 4.1. *Every bipartite trigraph is 2-clique-colorable.*

Proof. Let G be a bipartite trigraph, and let (A, B) be a partition of $V(G)$ into two strongly stable sets of G. Then every clique of G included in A or B is of size at most one, and is therefore not important. By Proposition 2.1 then, G is 2-clique-colorable. □

Proposition 4.2. *Every tame complement-bipartite trigraph that contains no heavy vertices is 2-clique-colorable.*

Proof. Let G be a tame complement-bipartite trigraph with no heavy vertices. Since $\overline{G}$ is bipartite, it contains no triangles and therefore no heavy vertices; consequently, G contains no light vertices. Thus, G is monogamous. If G is the null trigraph, then the result is immediate, so assume that $V(G) \neq \emptyset$. If $V(G)$ is a strong clique of G, then we assign the color 1 to one vertex of G, and the color 2 to all the other vertices of G, and we are done. So assume that $V(G)$ is not a strong clique of G. Let (A, B) be a partition of $V(G)$ into two strong cliques; since $V(G)$ is not a strong clique of G, we know that A and B are both non-empty, and that they are not strongly complete to each other. Note that all semi-adjacent pairs in G are between A and B. If G contains at least one semi-adjacent pair, then fix semi-adjacent vertices $a \in A$ and $b \in B$, and otherwise, fix strongly anti-adjacent vertices $a \in A$ and $b \in B$. Let A_b be the set of all strong neighbors of b in A, and let B_a be the set of all strong neighbors of a in B. Since G is monogamous, the choice of a and b guarantees that a is strongly anti-complete to $B \smallsetminus (\{b\} \cup B_a)$, and that b is strongly anti-complete to $A \smallsetminus (\{a\} \cup A_b)$.

Suppose first that $A_b = B_a = \emptyset$. Fix a clique K of G of size at least two, and suppose that either $K \subseteq \{a\} \cup (B \smallsetminus \{b\})$ or $K \subseteq \{b\} \cup (A \smallsetminus \{a\})$. By hypothesis, a is strongly anti-complete to $B \smallsetminus \{b\}$, and b is strongly anti-complete to $A \smallsetminus \{a\}$. Since K is a clique of size at least two, we deduce that either $K \subseteq B \smallsetminus \{b\}$ or $K \subseteq A \smallsetminus \{a\}$. In the former case, b is strongly complete to K, and in the latter case, a is strongly complete to K. This proves that no important clique of G is included in either $\{a\} \cup (B \smallsetminus \{b\})$ or $\{b\} \cup (A \smallsetminus \{a\})$, and Proposition 2.1 now implies that G is 2-clique-colorable.

Suppose now that at least one of A_b and B_a is non-empty; by symmetry, we may assume that $A_b \neq \emptyset$. By construction, $A_b \neq \emptyset$ is strongly complete to $\{b\} \cup (A \smallsetminus A_b)$, and b is strongly complete to $A_b \cup (B \smallsetminus \{b\})$; consequently, no important clique of G is included in either $\{b\} \cup (A \smallsetminus A_b)$ or $A_b \cup (B \smallsetminus \{b\})$. Proposition 2.1 applied to the partition $(\{b\} \cup (A \smallsetminus A_b), A_b \cup (B \smallsetminus \{b\}))$ of $V(G)$ now implies that G is 2-clique-colorable. $\square$

We next deal with line trigraphs. Theorem 2 of [1] and Theorem 5 of [2] independently imply the following.

Proposition 4.3. *[1, 2] Line graphs of bipartite graphs are* 2*-clique-colorable.*

As a corollary, we have the following.

Proposition 4.4. *Every line trigraph is* 2*-clique-colorable.*

Proof. Let G be a line trigraph, and let G_f be the full realization of G. By Proposition 4.3, G_f is 2-clique-colorable. Since G is a line trigraph, every clique of G of size at least three is strong, and consequently, every important clique of G is also an important clique of G_f. Thus, every clique-coloring of G_f is also a clique-coloring of G. The fact that G_f is 2-clique-colorable now implies that G is 2-clique-colorable. $\square$

We next handle complements of line trigraphs. Theorem 3 of [1] states that there exists a list $H_1, \dots, H_9$ of graphs, none of them bipartite, such that every graph H that is not isomorphic to any one of $H_1, \dots, H_9$, satisfies $\chi_C(\overline{L(H)}) \leq 2$. As an immediate corollary, we have the following.

Proposition 4.5. *[1] The complement of the line graph of a bipartite graph is* 2*-clique-colorable.*

Given a graph H, we say that a trigraph G is *H-free* provided that every realization of G is H-free, that is, provided that no realization of G contains (an isomorphic copy of) H as an induced subgraph. As usual, we denote by K_n the complete graph on n vertices. We denote by $K_1 \cup K_3$ the disjoint union of K_1 and K_3, and we denote by $2K_1 \cup K_2$ the four-vertex graph that contains exactly one edge. The *claw* is the complete bipartite graph $K_{1,3}$, and the *diamond* is the graph obtained from K_4 by deleting one edge. Note that $K_1 \cup K_3$ is the complement of the claw, and that $2K_1 \cup K_2$ is the complement of the diamond.

Proposition 4.6. *Let G be the complement of a line trigraph. Then G is both $(K_1 \cup K_3)$-free and $(2K_1 \cup K_2)$-free.*

Proof. It is routine to verify that every line graph of a bipartite graph is claw-free and diamond-free (this also follows from [18] and, independently, from Lemma 3.19 of [22]). By Proposition 3.5, every realization of a line trigraph is the line graph of a bipartite graph. This proves that every line trigraph is claw-free and diamond-free. The result now follows from the fact that $K_1 \cup K_3$ is the complement of the claw, and that $2K_1 \cup K_2$ is the complement of the diamond. □

Proposition 4.7. *Let G be the complement of a tame line trigraph, and assume that G contains no heavy vertices. Then G is 2-clique-colorable.*

Proof. Since $\overline{G}$ is a line trigraph, we know that all triangles in $\overline{G}$ are strong, and so $\overline{G}$ contains no heavy vertices; consequently, G contains no light vertices, and it follows that G is monogamous. If G contains no semi-adjacent pairs (that is, if G is a graph), then we are done by Proposition 4.5. It remains to consider the case when G contains at least one semi-adjacent pair, say uv. Since all triangles in $\overline{G}$ are strong, we know that no vertex in $V(G) \setminus \{u, v\}$ is complete to $\{u, v\}$ in $\overline{G}$; consequently, no vertex in $V(G)$ is anti-complete to $\{u, v\}$ in G. Since uv is a semi-adjacent pair in G, and since G is monogamous, we know that G contains no semi-adjacent pairs between $\{u, v\}$ and $V(G) \setminus \{u, v\}$. Let U be the set of all vertices in $V(G) \setminus \{u, v\}$ that are strongly adjacent to u and strongly anti-adjacent to v in G; let V be the set of all vertices in $V(G) \setminus \{u, v\}$ that are strongly adjacent to v and strongly anti-adjacent to u in G; and let W be the set of all vertices in $V(G) \setminus \{u, v\}$ that are strongly complete to $\{u, v\}$ in G. Then $V(G) = \{u, v\} \cup U \cup V \cup W$, and the sets $\{u, v\}, U, V, W$ are pairwise disjoint.

We first show that $\{v\} \cup U$ and $\{u\} \cup V$ are strongly stable sets. Since v and u are strongly anti-complete to U and V, respectively, it suffices to show that U and V are strongly stable sets. Suppose otherwise. By symmetry, we may assume that there exist adjacent vertices $u_1, u_2 \in U$. But then $\{u, u_1, u_2\}$ is a triangle, and v is anti-complete to $\{u, u_1, u_2\}$; consequently, $K_1 \cup K_3$ is a realization of $G[v, u, u_1, u_2]$, contrary to Proposition 4.6. Thus, $\{u\} \cup V$ and $\{v\} \cup U$ are strongly stable sets. In particular, if $W = \emptyset$, then G is bipartite, and we are done by Proposition 4.1. So from now on, we assume that $W \neq \emptyset$.

Next, we claim that each vertex in $U \cup W$ has at most one anti-neighbor in V, and that each vertex in $V \cup W$ has at most one anti-neighbor in U. Suppose otherwise. By symmetry, we may assume that some vertex $x \in U \cup W$ has two distinct anti-neighbors, call them v_1 and v_2, in V. Since $\{u\} \cup V$ is a strongly stable set, and since u is strongly complete to $U \cup W$, we easily deduce that $2K_1 \cup K_2$ is a realization of $G[u, x, v_1, v_2]$, contrary to Proposition 4.6. This proves our claim.

Let U_0 be the set of all vertices in U that are anti-complete to W, and let V_0 be the set of all vertices in V that are anti-complete to W. Suppose first that both U_0 and V_0 are non-empty. Since each vertex in $W \neq \emptyset$ has at most one anti-neighbor in U and at most one anti-neighbor in V, we deduce that $|U_0| = |V_0| = 1$, say $U_0 = \{u_0\}$ and $V_0 = \{v_0\}$, and that W is strongly complete to $(U \setminus \{u_0\}) \cup (V \setminus \{v_0\})$. Since W is also strongly complete to $\{u, v\}$, it follows that W is strongly complete to $\{u, v\} \cup (U \setminus \{u_0\}) \cup (V \setminus \{v_0\}) = V(G) \setminus (\{u_0, v_0\} \cup W)$. Since $W \neq \emptyset$, it follows that no important clique of G is included in $V(G) \setminus (\{u_0, v_0\} \cup W)$. In view of Proposition 2.1, it only remains to show that no important clique of G is included in $\{u_0, v_0\} \cup W$. Fix a clique K of G of size at least two such that $K \subseteq \{u_0, v_0\} \cup W$. If $K \subseteq \{u_0\} \cup W$, then u is strongly complete to K, and if $K \subseteq \{v_0\} \cup W$, then v is strongly complete to K, and in either case, K is not important. So suppose that $u_0, v_0 \in K$. Since K is a clique, it follows that u_0v_0 is an adjacent pair. We now fix $w \in W$, and we observe that $w - u - u_0 - v_0 - v - w$ is an odd hole in G, contrary to the fact that (by Proposition 3.6) G is Berge. Thus, no important clique of G is included in $\{u_0, v_0\} \cup W$, and it follows that G is 2-clique-colorable.

It remains to consider the case when at least one of U_0 and V_0 is empty; by symmetry, we may assume that $V_0 = \emptyset$, so that every vertex in V has a strong neighbor in W. Since u is strongly complete to $W \neq \emptyset$, it follows that every vertex in $\{u\} \cup V$ has a strong neighbor in W. Suppose that K is a clique of G such that $K \subseteq \{u, v\} \cup V$. Since $\{u\} \cup V$ is a strongly stable set, it follows that there exists some $v' \in \{u\} \cup V$ such that $K \subseteq \{v, v'\}$. We know that v' has a strong neighbor, call it w, in W. But now w is a strongly complete to K, and so K is not important. Thus, no important clique of G is included in $\{u, v\} \cup V$. Since u is strongly complete to $U \cup W$, we know that no important clique of G is included in $U \cup W$ either, and so Proposition 2.1 applied to the partition $(\{u, v\} \cup V, U \cup W)$ of $V(G)$ implies that G is 2-clique-colorable. □

Proposition 4.8. *Every double-split trigraph is 2-clique-colorable.*

Proof. Let G be a double-split trigraph, and let vertices $x_1, x_1', \dots, x_s, x_s'$, $y_1, y_1', \dots, y_t, y_t'$ (with $s, t \geq 2$) be as in the definition of a double-split trigraph. Using the definition of a double-split trigraph, we may assume by symmetry that x_1 is strongly complete to $\{y_1, \dots, y_t\}$ and strongly anti-complete to $\{y_1', \dots, y_t'\}$; then x_1' is strongly complete to $\{y_1', \dots, y_t'\}$ and strongly anti-complete to $\{y_1, \dots, y_t\}$. Further, we may assume that for all $i \in \{2, \dots, s\}$, x_i has at least one neighbor in $\{y_1, \dots, y_t\}$ (if not, we simply swap the roles of x_i and x_i'). In view of Proposition 2.1, it now suffices to show that no important clique of G is included in either $\{x_1, \dots, x_s\} \cup \{y'_1, \dots, y'_t\}$ or $\{x'_1, \dots, x'_s\} \cup \{y_1, \dots, y_t\}$. Fix a clique K of G of size at least two, and assume that either $K \subseteq \{x_1, \dots, x_s\} \cup \{y'_1, \dots, y'_t\}$ or $K \subseteq \{x'_1, \dots, x'_s\} \cup \{y_1, \dots, y_t\}$; we must show that K is not important. By construction, x_1 is strongly anti-complete to $\{x_2, \dots, x_s\} \cup \{y'_1, \dots, y'_t\}$, and x'_1 is strongly anti-complete to $\{x'_2, \dots, x'_s\} \cup \{y_1, \dots, y_t\}$; since K is a clique of size at least two, it follows that either $K \subseteq \{x_2, \dots, x_s\} \cup \{y'_1, \dots, y'_t\}$ or $K \subseteq \{x'_2, \dots, x'_s\} \cup \{y_1, \dots, y_t\}$. Since $\{x_2, \dots, x_s\}$ is a strongly stable set and K is a clique, we know that K contains at most one member of $\{x_2, \dots, x_s\}$; similarly, K contains at most one member of $\{x'_2, \dots, x'_s\}$. By symmetry, we may now assume that either $K \subseteq \{x_2, y'_1, \dots, y'_t\}$ or $K \subseteq \{x'_2, y_1, \dots, y_t\}$. Further, if $K \subseteq \{y_1, \dots, y_t\}$, then x_1 is strongly complete to K, and if $K \subseteq \{y'_1, \dots, y'_t\}$, then x'_1 is strongly complete to K, and in either case K is not important. Thus, we may assume that either $x_2 \in K \subseteq \{x_2, y'_1, \dots, y'_t\}$ or $x'_2 \in K \subseteq \{x'_2, y_1, \dots, y_t\}$. By hypothesis, x_2 has a neighbor in $\{y_1, \dots, y_t\}$; by symmetry, we may assume that x_2 is adjacent to y_1. Then x_2y_1 and $x'_2y'_1$ are strongly adjacent pairs, and $x_2y'_1$ and x'_2y_1 are strongly anti-adjacent pairs. Since K is a clique, we deduce that either $K \subseteq \{x_2, y'_2, \dots, y'_t\}$ or $K \subseteq \{x'_2, y_2, \dots, y_t\}$. In the former case, y_1 is strongly complete to K, and in the latter case, y'_1 is strongly complete to K. In either case, K is not important, and it follows that G is 2-clique-colorable. □

Lemma 4.9. *Let G be tame Berge trigraph that does not admit a balanced skew-partition. If G is a bipartite trigraph, a complement-bipartite trigraph, a line trigraph, the complement of a line trigraph, or a double-split trigraph, then either G is isomorphic to E_1 or E_2, or G is 2-clique-colorable.*

Proof. If G contains a heavy vertex, this follows from Lemma 3.11. Otherwise, the result follows from Propositions 4.1, 4.2, 4.4, 4.7, and 4.8. □

5 Decompositions

In this section, we show that odd and even 2-joins in a sense "preserve" the property of being 2-clique-colorable. We deal with the case when a trigraph G admits an odd or even 2-join (see Proposition 5.1), when $\overline{G}$ admits an odd 2-join (see Proposition 5.2), and when $\overline{G}$ admits an even 2-join (see Proposition 5.3). The results of this section essentially represent the induction step of the proof by induction of Theorem 3.1.

Proposition 5.1. *Let G be a trigraph that admits a 2-join (X_1, X_2) that is either odd or even, and let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of this 2-join. If (X_1, X_2) is odd, then let vertices a_1, b_1, a_2, b_2 and trigraphs G_1, G_2 be as in Proposition 3.3; and if (X_1, X_2) is even, then let vertices $a_1, b_1, c_1, a_2, b_2, c_2$*

and trigraphs G_1, G_2 be as in Proposition 3.4. If G_1 and G_2 are both 2-clique-colorable, then G is 2-clique-colorable.

Proof. Assume that G_1 and G_2 are 2-clique-colorable. Our first goal is to construct clique-colorings $q_1 : V(G_1) \to \{1, 2\}$ and $q_2 : V(G_2) \to \{1, 2\}$ of G_1 and G_2, respectively, such that $q_1(a_2) \neq q_2(a_1)$ and $q_1(b_2) \neq q_2(b_1)$.

Suppose first that (X_1, X_2) is odd. For each $i \in \{1, 2\}$, we fix a clique-coloring $q_i : V(G_i) \to \{1, 2\}$ of G_i; since $\{a_{3-i}, b_{3-i}\}$ is an important clique in G_i, we know that $q_i(a_{3-i}) \neq q_i(b_{3-i})$. By symmetry, we may assume that $q_1(a_2) = q_2(b_1)$. Since the codomain of both q_1 and q_2 is the two-element set $\{1, 2\}$, we deduce that $q_1(b_2) = q_2(a_1)$, $q_1(a_2) \neq q_2(a_1)$, and $q_1(b_2) \neq q_2(b_1)$.

Suppose now that (X_1, X_2) is even. For each $i \in \{1, 2\}$, we fix a clique-coloring $q_i : V(G_i) \to \{1, 2\}$ of G_i; since $\{a_{3-i}, c_{3-i}\}$ and $\{b_{3-i}, c_{3-i}\}$ are important cliques in G_i, we know that $q_i(a_{3-i}) \neq q_i(c_{3-i}) \neq q_i(b_{3-i})$. The fact that the codomain of q_i has only two elements now implies that $q_i(a_{3-i}) = q_i(b_{3-i})$. Since the codomain of both q_1 and q_2 is the two-element set $\{1, 2\}$, we may assume that $q_1(a_2) \neq q_2(a_1)$ and $q_1(b_2) \neq q_2(b_1)$.

Thus, we may assume that $q_1 : V(G_1) \to \{1, 2\}$ and $q_2 : V(G_2) \to \{1, 2\}$ are clique-colorings of G_1 and G_2, respectively, such that $q_1(a_2) \neq q_2(a_1)$ and $q_1(b_2) \neq q_2(b_1)$. Let $q : V(G) \to \{1, 2\}$ be given by

$$q(v) = \begin{cases} q_1(v) & \text{if} \quad v \in A_1 \cup B_1 \cup C_1 \\ q_2(v) & \text{if} \quad v \in A_2 \cup B_2 \cup C_2 \end{cases}$$

for all $v \in V(G)$. We need to show that q is a clique-coloring of G. Fix a clique K of G of size at least two, monochromatic with respect to q. We need to show that some vertex in $V(G) \smallsetminus K$ is strongly complete to K in G. Since K is a clique of G, we may assume by symmetry that one of the following holds:

(a) $K \subseteq A_1 \cup B_1 \cup C_1$;

(b) $K \subseteq A_1 \cup A_2$, $K \cap A_1 \neq \emptyset$, and $K \cap A_2 \neq \emptyset$.

Suppose first that (a) holds. Then K is a clique of G_1 of size at least two, monochromatic with respect to the clique-coloring q_1 of G_1. Thus, there exists some $w_1 \in V(G_1) \smallsetminus K$ such that w_1 is strongly complete to K in G_1. If $w_1 \in A_1 \cup B_1 \cup C_1$, then let $w = w_1$; if $w_1 = a_2$, then let w be an arbitrary vertex of A_2; and if $w_1 = b_2$, then let w be an arbitrary vertex of B_2. In any case, $w \in V(G) \smallsetminus K$ is strongly complete to K in G, and we are done.

Suppose now that (b) holds. Since $q_1(a_2) \neq q_2(a_1)$, and the codomain of both

q_1 and q_2 is the two-element set $\{1,2\}$, we deduce that $\{q_1(a_2), q_2(a_1)\} = \{1,2\}$, so that the codomain of q is $\{q_1(a_2), q_2(a_1)\}$. Since K is monochromatic with respect to q, we may assume by symmetry that $q[K] = \{q_1(a_2)\}$. Set $K_1 = (K \smallsetminus A_2) \cup \{a_2\}$. Then $q_1[K_1] = \{q_1(a_2)\}$. Furthermore, since K intersects both A_1 and A_2, K_1 is a clique of size at least two in G_1. Since q_1 is a clique-coloring of G_1, we know that some vertex $w_1 \in V(G_1) \smallsetminus K_1$ is strongly complete to K_1 in G. Since $a_2 \in K_1$, and since all the strong neighbors of a_2 in G_1 belong to A_1, we deduce that $w_1 \in A_1 \smallsetminus K_1$. It now follows that $w_1 \in V(G) \smallsetminus K$, and that w_1 is strongly complete to K in G. □

Proposition 5.2. *Let G be a trigraph that admits an odd 2-join with split $(A_1, B_1, C_1, A_2, B_2, C_2)$. Let vertices a_1, b_1, a_2, b_2 and trigraphs G_1, G_2 be as in Proposition 3.3, and assume that $\overline{G_1}$ and $\overline{G_2}$ are 2-clique-colorable. Then $\overline{G}$ is 2-clique-colorable.*

Proof. Note that C_2 is strongly complete to $A_1 \cup B_1 \cup C_1$ in $\overline{G}$, and C_1 is strongly complete to $A_2 \cup B_2 \cup C_2$ in $\overline{G}$. If C_1 and C_2 are both non-empty, this implies that no important clique of $\overline{G}$ is included in either $A_1 \cup B_1 \cup C_1$ or $A_2 \cup B_2 \cup C_2$, and so by Proposition 2.1, $\overline{G}$ is 2-clique-colorable. So assume that at least one of C_1 and C_2 is empty; by symmetry, we may assume that $C_2 = \emptyset$. Fix a clique-coloring $q_1 : V(G_1) \to \{1,2\}$ of $\overline{G_1}$, and define $q : V(G) \to \{1,2\}$ by setting

$$q(v) = \begin{cases} q_1(v) & \text{if} \quad v \in A_1 \cup B_1 \cup C_1 \\ q_1(a_2) & \text{if} \quad v \in A_2 \\ q_1(b_2) & \text{if} \quad v \in B_2 \end{cases}$$

for all $v \in V(G)$. To show that q is a clique-coloring of $\overline{G}$, we fix a clique K of $\overline{G}$ of size at least two, monochromatic with respect to q; we need to show that some vertex in $V(G) \smallsetminus K$ is strongly complete to K in $\overline{G}$.

Suppose first that K intersects both A_2 and B_2. Set $K_1 = (K \smallsetminus (A_2 \cup B_2)) \cup \{a_2, b_2\}$. Then K_1 is a clique in $\overline{G_1}$ of size at least two, and by construction, $q_1[K_1] = q[K]$. Since K is monochromatic with respect to q, it follows that K_1 is monochromatic with respect to q_1. Since q_1 is a clique-coloring of $\overline{G_1}$, we know that some vertex $w_1 \in V(G_1) \smallsetminus K_1$ is strongly complete to K_1 in $\overline{G_1}$. Since $a_2, b_2 \in K_1$, we know that $w_1 \notin \{a_2, b_2\}$. Thus, $w_1 \in V(G) \smallsetminus K$, and it is easy to see that w_1 is strongly complete to K in $\overline{G}$.

Suppose next that K intersects exactly one of A_2 and B_2; by symmetry, we may assume that $K \cap A_2 \neq \emptyset$ and $K \cap B_2 = \emptyset$. Then $K \subseteq A_1 \cup B_1 \cup C_1 \cup A_2$. If $K \subseteq A_2$, then $B_1 \neq \emptyset$ is strongly complete to K in $\overline{G}$, and we are done. So assume that $K \smallsetminus A_2 \neq \emptyset$. Set $K_1 = (K \smallsetminus A_2) \cup \{a_2\}$. Then K_1 is a clique of size at least two in $\overline{G_1}$, and by construction, $q_1[K_1] = q[K]$. Since K is monochromatic with respect to q, it follows that K_1 is monochromatic with respect to q_1. Since q_1 is a clique-coloring of $\overline{G_1}$, there exists a vertex $w_1 \in V(G_1) \setminus K_1$ that is strongly complete to K_1 in $\overline{G}_1$. Since $a_2 \in K_1$ and a_2b_2 is a semi-adjacent pair in $\overline{G_1}$, we know that $w_1 \notin \{a_2, b_2\}$. Thus, $w_1 \in A_1 \cup B_1 \cup C_1$. It follows that $w_1 \in V(G) \setminus K$, and it is easy to see that w_1 is strongly complete to K in $\overline{G}$.

It remains to consider the case when K intersects neither A_2 nor B_2. Then K is a clique of size at least two in $\overline{G_1}$, monochromatic with respect to the clique-coloring q_1 of $\overline{G}_1$. Thus, some vertex $w_1 \in V(G_1) \setminus K$ is strongly complete to K in $\overline{G_1}$. If $w_1 \in A_1 \cup B_1 \cup C_1$, then let $w = w_1$; if $w_1 = a_2$, then let w be any vertex in A_2; and if $w_1 = b_2$, then let w be any vertex in B_2. Then $w \in V(G) \setminus K$, and w is strongly complete to K in $\overline{G}$. □

Proposition 5.3. *Let G be a trigraph that admits an even 2-join. Then $\overline{G}$ is 2-clique-colorable.*

Proof. Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of an even 2-join of G. Since this 2-join of G is even, we know that for each $i \in \{1, 2\}$, A_i is strongly anti-complete to B_i in G, and therefore, that A_i is strongly complete to B_i in $\overline{G}$. It now follows that $B_1 \neq \emptyset$ is strongly complete to $A_1 \cup A_2 \cup C_2$ in $\overline{G}$, and that $A_2 \neq \emptyset$ is strongly complete to $B_1 \cup B_2 \cup C_1$ in $\overline{G}$; consequently, no important clique of $\overline{G}$ is included in either $A_1 \cup A_2 \cup C_2$ or $B_1 \cup B_2 \cup C_1$, and the result follows from Proposition 2.1. □

6 Proof of the main theorem

We are finally ready to prove Theorem 3.1 (restated below for the reader's convenience). We remind the reader that E_1 is the trigraph with vertex-set $\{a, b, c\}$ in which ab is a strongly adjacent pair, and ac, bc are semi-adjacent pairs, and that E_2 is the trigraph with vertex-set $\{a, b, c, d\}$ in which ab, cd are strongly adjacent pairs, ac, bc are semi-adjacent pairs, and ad, bd are strongly anti-adjacent pairs.

Theorem 3.1. *Every tame Berge trigraph that does not admit a balanced skew-partition, and that is isomorphic to neither E_1 nor E_2, is 2-clique-colorable.*

Proof. Let $\mathcal{B}$ be the class of all tame Berge trigraphs that do not admit a balanced skew-partition. We claim that all trigraphs in $\mathcal{B}$ are either isomorphic to one of E_1 and E_2, or 2-clique-colorable. Fix $G \in \mathcal{B}$, and assume inductively that the claim holds for trigraphs in $\mathcal{B}$ that have fewer vertices than G does. We need to show that the claim holds for G. In view of Theorem 3.10 and Lemma 4.9, we may assume that G or $\overline{G}$ admits a proper

2-join. (Note that $\overline{G} \in \mathcal{B}$, as $\mathcal{B}$ is closed under complementation.) To complete the induction, we will prove the following stronger statement: both G and $\overline{G}$ are 2-clique-colorable. We now have symmetry between G and $\overline{G}$, and so we may assume that G admits a proper 2-join.

Let (X_1, X_2) be a proper 2-join of G, and let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be its split. By Proposition 3.2, the 2-join (X_1, X_2) of G is either odd or even. If (X_1, X_2) is odd, then let vertices a_1, b_1, a_2, b_2 and trigraphs G_1, G_2 be as in Proposition 3.3; and if (X_1, X_2) is even, then let vertices $a_1, b_1, c_1, a_2, b_2, c_2$ and trigraphs G_1, G_2 be as in Proposition 3.4. By Propositions 3.3 and 3.4, we know that $G_1, G_2 \in \mathcal{B}$; since the class $\mathcal{B}$ is closed under complementation, it follows that that $\overline{G_1}, \overline{G_2} \in \mathcal{B}$. Further, using Propositions 3.3 and 3.4, we easily deduce that $6 \leq |V(G_i)| < |V(G)|$ for each $i \in \{1, 2\}$. Since E_1 and E_2 have fewer than six vertices, we know that none of $G_1, G_2, \overline{G_1}, \overline{G_2}$ is isomorphic to either E_1 or E_2. The induction hypothesis now guarantees that $G_1, G_2, \overline{G_1}, \overline{G_2}$ are all 2-clique-colorable. The fact that G is 2-clique-colorable now follows from Proposition 5.1, and the fact that $\overline{G}$ is 2-clique-colorable follows from Propositions 5.2 and 5.3. This completes the argument. □

Corollary 6.1. *Every monogamous Berge trigraph that does not admit a balanced skew-partition is* 2*-clique-colorable.*

Proof. Since the trigraphs E_1 and E_2 are not monogamous, this is an immediate consequence of Theorem 3.1. □

Theorem 1.2, stated in the Introduction and restated below, is simply a special case of Corollary 6.1.

Theorem 1.2. *Every perfect graph that does not admit a balanced skew-partition is* 2*-clique-colorable.*

Proof. Let G be a perfect graph that does not admit a balanced skew-partition. Since G is perfect, it is Berge (as discussed in the Introduction, this is the "easy direction" of the Strong Perfect Graph Theorem [9]). Since every graph is a monogamous trigraph, Corollary 6.1 implies that G is 2-clique-colorable. □

In view of Theorem 1.2, one might ask whether there are any perfect graphs with no balanced skew-partition that do not belong to any class of graphs known to be 2-clique-colorable (or at least k-clique-colorable, for some fixed constant k) from previous results about clique-coloring [1, 2, 12, 13, 14]. The answer to this question is positive. It is routine to prove that double-split graphs are perfect, and Trotignon [23] showed that double-split graphs do not admit a balanced skew-partition (see Lemma 4.5 of [23]). It is easy to construct a double-split graph that contains all the following graphs and their complements as induced subgraphs: the claw, the diamond, and the net (the *net* is the 6-vertex graph that consists of a triangle and three vertex-disjoint pendant edges; clearly, the bull is an induced subgraph of the net). Note that such a double-split graph is neither a line graph nor the complement of a line graph, nor is it a comparability graph (here, we use the well-known and easy to check fact that line graphs are claw-free [3], and comparability graphs are net-free [15]). Furthermore, double-split graphs are not generalized split graphs (this follows from the fact that if G is a generalized split graph, then either one of $G, \overline{G}$ is a bipartite graph, or one of $G, \overline{G}$ admits a clique-cutset, whereas no double-split graph has this property). Thus, a double-split graph of the sort considered above (i.e. a double-split graph that contains the claw and its complement, the diamond and its complement, and the net and its complement as induced subgraphs) does not belong to any class shown to have a bounded clique-chromatic number in [1, 2, 12, 13, 14].

It may also be worth pointing out that Gravier et al. [16] showed that for all $k \geq 2$, graphs that do not contain an induced $(k+1)$-edge path are k-clique-colorable, and it is easy to see that double-split graphs contain no induced 5-edge paths. However, "path-double-split graphs" (introducted in [23]) are perfect graphs with no balanced skew-partition, and they can contain arbitrarily long induced paths. A *path-double-split graph* is any graph obtained from a double-split graph by subdividing each edge $x_i x_i'$ an even (possibly zero) number of times (thus, each edge $x_i x_i'$ becomes an induced odd path; here, vertices $x_1, x_1' \dots, x_s, x_s'$ are as in the definition of a double-split graph). The reader can verify that path-double-split graphs are perfect, and by Lemma 4.5 of [23], they do not admit a balanced skew-partition. Clearly, path-double-split graphs can contain arbitrarily long even cycles as induced subgraphs, and it is easy to construct a path-double-split graph that contains all the following as induced subgraphs: the claw and its complement, the diamond and its complement, the net and its complement, and an arbitrarily long even cycle (and therefore an arbitrarily long path).

7 Open problems

In this section, we discuss some open problems related to the results of this paper. First, it remains an open problem to determine whether there exists a constant c such that every perfect graph is c-clique colorable. If one were to attack this problem using the decomposition theorems for Berge (tri)graphs from [4, 9, 11], one would inevitably be confronted with the problem of dealing with balanced skew-partitions, which seems to be a rather daunting task. With this in mind, we propose a couple of potentially easier problems. We remind the reader that a class of graphs is said to be *hereditary* if it is closed under isomorphism and induced subgraphs. A *clique-cutset* of a graph G is a (possibly empty) clique K of G such that $G \smallsetminus K$ (the graph obtained from G by deleting K) is disconnected. We propose the following two problems concerning clique-cutsets and clique-coloring (the second question is a special case of the first, restricted to perfect graphs).

Question 7.1. *Let c be a positive integer, and let $\mathcal{G}$ be a hereditary class of graphs. If all graphs in $\mathcal{G}$ are either c-clique-colorable or admit a clique-cutset, then must there exist a constant d such that every graph in $\mathcal{G}$ is d-clique-colorable?*

Question 7.2. *Let c be a positive integer, and let $\mathcal{G}$ be a hereditary class of perfect graphs. If all graphs in $\mathcal{G}$ are either c-clique-colorable or admit a clique-cutset, then must there exist a constant d such that every graph in $\mathcal{G}$ is d-clique-colorable?*

Note that the graph $C_9^{\triangle}$ (defined in the Introduction) admits a clique-cutset, and all of its proper induced subgraphs are 2-clique-colorable. However, $C_9^{\triangle}$ itself is not 2-clique-colorable. Thus, if one hopes to obtain positive answers to Questions 7.1 and 7.2, one cannot set $d = c$.

Finally, we repeat the question raised in section 2.

Question 7.3. *Is it true that for every monogamous trigraph G, one has that $\chi_C(G) = \max\{\chi_C(\widetilde{G}) \mid \widetilde{G}$ is a realization of $G\}$?*

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