Coloring Bull-Free Perfect Graphs

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Abstract

A graph G is *perfect* if for every induced subgraph H of G, the chromatic number of H equals the size of the largest complete subgraph of H. A *bull* is a graph on five vertices consisting of a triangle and two vertex-disjoint pendant edges. A graph is said to be *bull-free* if none of its induced subgraphs is a bull. In [C.M.H. de Figueiredo and F. Maffray, "Optimizing bull-free perfect graphs", *SIAM J. Discrete Math*, 18 (2004), 226-240], de Figueiredo and Maffray gave polynomial time combinatorial algorithms that solve the following four optimization problems for weighted bull-free perfect graphs with integer weights: the maximum weighted clique problem; the maximum weighted stable set problem; the minimum weighted coloring problem; and the minimum weighted clique covering problem. In this paper, we give faster combinatorial algorithms that solve the same four problems. The running time of our algorithms for finding a maximum weighted clique and a maximum weighted stable set in a weighted bull-free perfect graph with integer weights is $O(n^6)$, and the running time of our algorithms for finding a minimum weighted coloring and a minimum weighted clique covering in such a graph is $O(n^8)$, where n is the number of vertices of the input graph.

1 Introduction

All graphs in this paper are simple and finite. The vertex set and the edge set of a graph G are denoted by V_G and E_G, respectively. Given a graph G, we denote by $\overline{G}$ the complement of G; a class $\mathcal{G}$ of graphs is said to be *self-complementary* if for every graph $G \in \mathcal{G}$, we have that $\overline{G} \in \mathcal{G}$. A *clique* in G is a set of pairwise adjacent vertices of G, and a *stable set* in G is a set of pairwise non-adjacent vertices in G. The clique number of G, written $\omega(G)$, is the maximum number of vertices in a clique in G. A *coloring* of G is a partition of V_G into stable sets; the *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest number of stable sets needed to partition the vertex set of G. A graph G is said to be *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. A *hole* in a graph G is an induced cycle of length at least four in G. An *anti-hole* in G is an induced subgraph of G whose complement is a hole in $\overline{G}$. A hole or anti-hole is *odd* provided that it contains an odd number of vertices. A graph is said to be *Berge* if it contains no odd holes and no odd anti-holes. In this paper, we use (and state below) two results about perfect graphs: the "weak perfect graph theorem" (1.1) due to Lovász, and the "strong perfect graph theorem" (1.2) due to Chudnovsky, Robertson, Seymour, and Thomas.

1.1 (Lovász [19]). *The class of perfect graphs is self-complementary, that is, if a graph G is perfect then so is its complement $\overline{G}$.*

1.2 (Chudnovsky, Robertson, Seymour, and Thomas [9]). *A graph G is perfect if and only if it is Berge.*

The weak perfect graph theorem 1.1 and the strong perfect graph theorem 1.2 were originally conjectured by Berge in the 1960's [1]. As the class of Berge graphs is self-complementary, the weak perfect graph theorem 1.1 is an immediate consequence of the strong perfect graph theorem 1.2; however, 1.1 was proven long before 1.2, and so we state the two theorems separately.

There have been a number of results on the algorithmic aspects of perfect graphs. A few years ago, Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [7] found a polynomial time recognition algorithm for Berge graphs; by the strong perfect graph theorem 1.2, this is in fact a recognition algorithm for perfect graphs. Another important result was a polynomial time coloring algorithm for perfect graphs, due to Grötschel, Lovász, and Schrijver [17]. However, this coloring algorithm is based on the ellipsoid method, and it remains an open problem to find a polynomial time combinatorial coloring algorithm for perfect graphs. So far, research in this direction has focused on special cases of perfect graphs, and one of the classes that has received attention is the class of "bull-free" perfect graphs.

A *bull* is a graph with vertex set $\{x_1, x_2, x_3, y_1, y_2\}$ and edge set $\{x_1x_2, x_2x_3, x_3x_1, x_1y_1, x_2y_2\}$; a graph is said to be *bull-free* provided that it does not contain a bull as an induced subgraph. We note that the complement of a bull is again a bull; this, together with the weak perfect graph theorem 1.1 implies that the class of bull-free perfect graphs is self-complementary, and we state this result below for future reference.

1.3. *The class of bull-free perfect graphs is self complementary, that is, the complement of a bull-free perfect graph is bull-free and perfect.*

Bull-free graphs were originally studied in the context of perfect graphs. Thus, long before the proof of the strong perfect graph theorem (1.2) was announced, Chvátal and Sbihi [10] proved that bull-free Berge graphs are perfect. Similarly, Reed and Sbihi [21] gave a polynomial time recognition algorithm for bull-free Berge graphs before the recognition algorithm for Berge graphs [7] was found. Finally, de Figueiredo and Maffray [14] found polynomial time combinatorial algorithms for four optimization problems for weighted bull-free perfect graphs: the "maximum weighted clique problem"; the "maximum weighted stable set problem"; the "minimum weighted coloring problem"; and the "minimum weighted clique covering problem" (we describe these four problems in detail below). Recently, in a series of papers, Chudnovsky [3, 4, 5, 6] gave a structure theorem for bull-free graphs, and together with Chudnovsky, we obtained a structure theorem for bull-free perfect graphs [8]. In the present paper, we use the results from [8] to give polynomial time combinatorial algorithms, faster than the ones given in [14], that solve the four optimization problems mentioned above.

We now need some definitions. In this paper, a *weighted graph* is a graph G such that every vertex v of G is assigned a positive integer *weight*, denoted by $w_G(v)$. Given a set $S \subseteq V_G$, the *weight* of S, denoted by $w_G(S)$, is the sum of the weights of the vertices in S; the weight of the empty set is assumed to be zero. Unless stated otherwise, an induced subgraph H of a weighted graph G is assumed to inherit the weights from G, that is, it is assumed that $w_H(v) = w_G(v)$ for all $v \in V_H$. A *maximum weighted clique* (respectively: *maximum weighted stable set*) in G is a clique (respectively: stable set) that has the maximum weight among all the cliques (respectively: stable sets) in G. We denote by $W(G)$ the maximum weight of a clique in G. We now describe the four optimization problems mentioned above. First, the *maximum weighted clique problem* (respectively: *maximum weighted stable set problem*) is the problem of finding a maximum weighted clique (respectively: maximum weighted stable set) in a weighted graph. Next, the *minimum weighted coloring problem* is the problem of finding stable sets $S_1, ..., S_t$ in a weighted graph G, together with positive integers $\lambda_1, ..., \lambda_t$, such that $\Sigma_{S_i \ni v} \lambda_i \geq w_G(v)$ for all $v \in V_G$, and with the property that $\Sigma_{i=1}^{t} \lambda_i$ is minimum. Finally, the *minimum weighted clique covering problem* is the problem of finding cliques $C_1, ..., C_t$ in a weighted graph G, together with positive integers $\lambda_1, ..., \lambda_t$, such that $\Sigma_{C_i \ni v} \lambda_i \geq w_G(v)$ for all $v \in V_G$, and with the property that $\Sigma_{i=1}^{t} \lambda_i$ is minimum.

Clearly, any maximum weighted clique in a graph G is a maximum weighted stable set in $\overline{G}$. Thus, since the class of bull-free perfect graphs is self-complementary (by 1.3), it is easy to see that the maximum weighted clique problem and the maximum weighted stable set problem are equivalent for this class of graphs in the following sense: any algorithm A that finds a maximum weighted clique in any bull-free perfect graph in $O(n^k)$ time can be "turned into" an algorithm that finds a maximum weighted stable set in any bull-free perfect graph in $O(n^{\max\{k,2\}})$ time, where n is the number of vertices of the input graph G; indeed, we first take the complement of G, which takes $O(n^2)$ time, and then we run the algorithm A on $\overline{G}$ to obtain a maximum weighted clique C in $\overline{G}$, which is clearly a maximum weighted stable set in G. The reverse also holds: any algorithm that finds a maximum weighted stable set in any bull-free perfect graph in $O(n^k)$ time can be "turned into" an algorithm that finds a maximum weighted clique in any bull-free perfect graph in $O(n^{\max\{k,2\}})$ time. Clearly, the minimum weighted coloring and the minimum weighted clique covering problems are also equivalent for the class of bull-free perfect graphs in this same sense.

Further, it follows implicitly from the proof of Corollary 67.5c from [22] that if $\mathcal{G}$ is a class of perfect graphs, closed under taking induced subgraphs, and A is any algorithm that can find a maximum weighted clique and stable set in each graph in $\mathcal{G}$ in $O(n^k)$ time, then the algorithm A can be used to obtain an algorithm B that can find a minimum weighted coloring in each graph in $\mathcal{G}$ in $O(n^{k+2})$ time, where n is the number of vertices of the input graph. The details can be found in [22], but let us provide a brief outline of the argument here. Suppose we are given a weighted graph $G \in \mathcal{G}$ on n vertices. The first step is to find a stable set S in G that intersects each maximum weighted clique in G. To do this, we first find a maximum weighted clique in G, and then we recursively extend a list $K_1, ..., K_t$ of maximum weighted cliques in G, as follows. We consider the graph $G[K_1 \cup ... \cup K_t]$ (the subgraph of G induced by $K_1 \cup ... \cup K_t$), with reassigned weights: for each vertex $v \in K_1 \cup ... \cup K_t$, the weight assigned to v in $G[K_1 \cup ... \cup K_t]$ is the number of cliques $K_1, ..., K_t$ that v lies in. We then find a maximum weighted stable set S in $G[K_1 \cup ... \cup K_t]$; since G is perfect, this stable set intersects each of $K_1, ..., K_t$. We then find a maximum weighted clique in the graph $G \setminus S$ (the graph obtained from G by deleting all the vertices in S); if this clique has the same weight as the cliques $K_1, ..., K_t$, then we add this clique to the end of the list $K_1, ..., K_t$, and we repeat the process, and otherwise, S is a stable set that intersects each maximum weighted clique in G. This process involves making at most $O(n)$ calls to the algorithm that finds a maximum weighted clique or stable set in G or its induced subgraphs (possibly with modified weights). The stable set S that intersects each maximum weighted clique in G then becomes a color class of the desired coloring of G (the algorithm determines the appropriate weight λ associated with the color class S). The process is then repeated for an induced subgraph of G (with reassigned weights); after at most $O(n)$ iterations, we obtain a minimum weighted coloring.

All this implies that, in order to solve the four optimization problems for the class of bull-free perfect graphs in polynomial time, it suffices to construct an algorithm that solves the maximum weighted clique problem in polynomial time.

The algorithm from [14] finds a maximum weighted clique in a weighted bull-free perfect graph in time $O(n^5m^3)$, where n is the number of vertices and m is the number of edges of the input graph, and it relies on the argument outlined above to solve the remaining three optimization problems. In the present paper, we give a polynomial time algorithm that solves the maximum weighted clique problem for weighted bull-free perfect graphs in $O(n^6)$ time, where n is the number of vertices of the input graph. Relying on the argument above, this yields an algorithm for finding a maximum weighted stable set in a weighted bull-free perfect graph in $O(n^6)$ time, as well as algorithms for solving the minimum weighted coloring problem and the minimum weighted clique covering problem in such a graph in $O(n^8)$ time, where n is the number of vertices of the input graph.

2 The Main Decomposition Theorem and a General Outline of the Paper

We begin with some definitions. Let G be a graph. Given a non-empty set $X \subseteq V_G$, we denote by $G[X]$ the subgraph of G induced by X. Given a set $S \subsetneqq V_G$, we denote by $G \smallsetminus S$ the graph $G[V_G \smallsetminus S]$, and given a vertex $v \in V_G$, we denote by $G \smallsetminus v$ the graph $G \smallsetminus \{v\}$. We say that a vertex $v \in V_G$ is *complete* (respectively: *anti-complete*) to a set $S \subseteq V_G \smallsetminus \{v\}$ provided that v is adjacent (respectively: non-adjacent) to every vertex in S; we say that v is *mixed* on S provided that v is neither complete nor anti-complete to S. Given disjoint sets $A, B \subseteq V_G$, we say that A is *complete* (respectively: *anti-complete*) to B provided that every vertex in A is complete (respectively: anti-complete) to B. A non-empty set $S \subsetneqq V_G$ is said to be a *homogeneous set* in G provided that no vertex in $V_G \smallsetminus S$ is mixed on S; a homogeneous set S in G is said to be *proper* if it contains at least two vertices. Given disjoint non-empty sets $A, B \subseteq V_G$, we say that (A, B) is a *homogeneous pair* in G provided that no vertex in $V_G \smallsetminus (A \cup B)$ is mixed on A, and no vertex in $V_G \smallsetminus (A \cup B)$ is mixed on B. If (A, B) is a homogeneous pair in $V_G \smallsetminus (A \cup B)$, and C is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are complete to A and anti-complete to B, D is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are complete to B and anti-complete to A, E is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are complete to $A \cup B$, and F is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are anti-complete to $A \cup B$, then we say that (A, B, C, D, E, F) is the *partition* of G associated with the homogeneous pair (A, B). A homogeneous pair (A, B) in G is said to be *reducible* provided that the associated partition (A, B, C, D, E, F) of G satisfies the following:

- either
    - $|B| \geq 3$, or

- $|B| = 2$ and there exist distinct vertices $a, a' \in A$ such that a and a' are both mixed on B;

- C and D are both non-empty.

We observe that if (A, B) is a reducible homogeneous pair in a graph G, then $|A \cup B| \geq 4$; furthermore, if (A, B) is a reducible homogeneous pair in G, then (A, B) is also a reducible homogeneous pair in $\overline{G}$, however, (B, A) need not be reducible in either G or $\overline{G}$. We remark that homogeneous pairs were first introduced by Chvátal and Sbihi in [10]; in that paper, they used homogeneous pairs (and other tools) to prove that every bull-free Berge graph is perfect. Reducible homogeneous pairs are introduced in the present paper for the first time.

In this paper, a *directed graph* is an ordered pair $\vec{G} = (V_G, A_G)$, where V_G is a non-empty set (called the *vertex set* of $\vec{G}$), and A_G (called the *arc set* of G) is an irreflexive, asymmetric binary relation on V_G; members of V_G are called the *vertices* of the directed graph $\vec{G}$, and members of A_G are called the *arcs* of $\vec{G}$. A directed graph $\vec{G} = (V_G, A_G)$ is a said to be *transitive* provided that for all $u, v, w \in V_G$, if $(u, v), (v, w) \in A_G$ then $(u, w) \in A_G$. A directed graph $\vec{G}$ is said to be an *orientation* of a graph G provided that:

- the vertex sets of the directed graph $\vec{G}$ and the graph G are identical;

- for all adjacent vertices u and v of G, exactly one of (u, v) and (v, u) is an arc of $\vec{G}$;

- for all non-adjacent vertices u and v of G, (u, v) is not an arc of $\vec{G}$.

A graph G is said to be *transitively orientable* provided that some orientation of G is a transitive directed graph.

The first goal of the present paper is to use the results of [8] to prove a decomposition theorem for bull-free perfect graphs (2.1), as well as a certain result about reducible homogeneous pairs (2.2), both of which we state below.

2.1. *Let G be a bull-free perfect graph. Then at least one of the following holds:*

- *G or $\overline{G}$ is transitively orientable;*

- *G contains a proper homogeneous set;*

- *G contains a reducible homogeneous pair.*

2.2. *Let G be a bull-free perfect graph that does not contain a proper homogeneous set, and let (A, B) be a reducible homogeneous pair in G. Then $G[A]$ and $G[B]$ are both transitively orientable.*

We remark that a number of other structural results about bull-free perfect graphs have been obtained (see, for instance, [10], [14], [15], and [21]). 2.1 is a new decomposition theorem for this class of graphs, and it will be used for the purposes of our algorithm. We remark that 2.2 could be proven using a result from [15] (see Lemma 2 from [15]), however, in the present paper, we prove a slightly more general theorem (see 5.2), and then derive 2.1 as an immediate consequence.

Our algorithm (called MWCLIQUE) for finding a maximum weighted clique in a weighted bull-free perfect graph is based on the two theorems stated above. The algorithm is described in detail in section 8, but let us give a brief outline here. (We note that the references to the previously known algorithms that the algorithm MWCLIQUE uses are all included in section 8, and we omit them here.) Suppose we are given a weighted bull-free perfect graph G. If the first outcome of 2.1 holds, then we win because there are known rapid algorithms for recognizing transitively orientable graphs and for finding transitive orientations of such graphs, and there are also known rapid algorithms for finding maximum weighted cliques and maximum weighted stable sets in weighted transitive directed graphs. If the second outcome holds, then we also win because there are known fast algorithms for finding homogeneous sets in a graph, and this allows us to make recursive calls to the algorithm MWCLIQUE to find maximum weighted cliques in two induced subgraphs of G (with slightly modified weights), which can then easily be combined to obtain a maximum weighted clique in G. So assume that neither G nor $\overline{G}$ is transitively orientable, and that G does not contain a proper homogeneous set. Then by 2.1, G contains a reducible homogeneous pair (A, B). We describe how to detect reducible homogeneous pairs in bull-free perfect graphs with no proper homogeneous set (see section 6), and we also describe how to "recover" a maximum weighted clique in G once we know maximum weighted cliques in four smaller graphs (see section 7); however, by 2.2, two of these four graphs (namely, $G[A]$ and $G[B]$) are transitively orientable, and so we need not make recursive calls to MWCLIQUE for these two graphs, which guarantees the polynomiality of our algorithm (see section 9 for a detailed complexity analysis).

We complete this section by giving an outline of the paper. In section 3, we introduce objects, called "trigraphs," which are a generalization of graphs, and we generalize certain graph theoretic concepts to trigraphs. While in a graph, two distinct vertices can be either adjacent or non-adjacent, in a trigraph, there are three options: a pair of distinct vertices can be adjacent, anti-adjacent, or semi-adjacent; semi-adjacent pairs can conveniently be thought of as having undecided adjacency. Every graph can be seen as a trigraph in a natural way: graphs are trigraphs in which no two vertices are semi-adjacent. We note here that we do not have a natural way to define a "trigraph coloring," and so we do not define "perfect trigraphs"; there is, however, a natural way to define a "Berge trigraph," as well as a "bull-free trigraph," and we do this in section 3. We remark that the structure theorem from [8] is in fact a structure theorem for bull-free Berge trigraphs; since every graph can be seen as a trigraph (and in particular, every bull-free Berge graph can be seen as a bull-free Berge trigraph), this is implicitly a structure theorem for bull-free Berge graphs (and via the strong perfect graph theorem 1.2, it is a structure theorem for bull-free perfect graphs). In section 4, we introduce directed trigraphs, we generalize the definition of a transitively orientable graph to trigraphs, and we prove a few results about transitively orientable trigraphs. In section 5, we use the results of [8] and of section 4 of the present paper to derive a decomposition theorem for bull-free Berge trigraphs (5.1), as well as a certain result about reducible homogeneous pairs in trigraphs (5.2); we note that 5.1 and 5.2 are trigraph analogues of, respectively, 2.1 and 2.2, stated above, and that 2.1 and 2.2 follow immediately from 5.1 and 5.2 (see the comment following the statements of 5.1 and 5.2 at the beginning of section 5). We note here that our algorithm deals only with graphs (and not with trigraphs); however, in order to prove 2.1 and 2.2, we need to use certain results about trigraphs, and this is the reason why we discuss trigraphs in this paper. In section 6, we give an algorithm that, given a graph G that does not contain a proper homogeneous set, either finds a reducible homogeneous pair in G, or determines that G does not contain one. In section 7, we explain how to obtain a maximum weighted clique in a graph that contains a proper homogeneous set or a reducible homogeneous pair once we have obtained maximum weighted cliques in certain smaller graphs. In section 8, we describe the algorithm MWCLIQUE that, given a weighted bull-free perfect graph G, finds a maximum weighted clique in G. In section 9, we perform a complexity analysis. Finally, in section 10, we discuss the reasons why the algorithm MWCLIQUE is faster than the algorithm from [14].

3 Trigraphs

A trigraph G is an ordered pair (V_G, θ_G), where V_G is a non-empty finite set, called the *vertex set* of G, and $\theta_G : V_G \times V_G \to \{-1, 0, 1\}$ is a map, called the *adjacency function* of G, satisfying the following:

- for all $v \in V_G$, $\theta_G(v, v) = 0$;
- for all $u, v \in V_G$, $\theta_G(u, v) = \theta(v, u)$;
- for all $u \in V_G$, there exists at most one $v \in V_G \setminus \{u\}$ such that $\theta_G(u, v) = 0$.

Members of V_G are called the *vertices* of G. Let $u, v \in V_G$ be distinct. We say that uv is a *strongly adjacent pair*, or that u and v are *strongly adjacent*, or that u is *strongly adjacent* to v, or that u is a *strong neighbor* of v, provided that $\theta_G(u,v) = 1$. We say that uv is a *strongly anti-adjacent pair*, or that u and v are *strongly anti-adjacent*, or that u is *strongly anti-adjacent* to v, or that u is a *strong anti-neighbor* of v, provided that $\theta_G(u,v) = -1$. We say that uv is a *semi-adjacent pair*, or that u and v are *semi-adjacent*, or that u is *semi-adjacent* to v, provided that $\theta_G(u,v) = 0$. (Note that we do not say that a vertex $w \in V_G$ is semi-adjacent to itself even though $\theta_G(w,w) = 0$.) If uv is a strongly adjacent pair or a semi-adjacent pair, then we say that uv is an *adjacent pair*, or that u and v are *adjacent*, or that u is *adjacent* to v, or that u is a *neighbor* of v. If uv is a strongly anti-adjacent pair or a semi-adjacent pair, then we say that uv is an *anti-adjacent pair*, or that u and v are *anti-adjacent*, or that u is *anti-adjacent* to v, or that u is an *anti-neighbor* of v. Thus, if uv is a semi-adjacent pair, then uv is simultaneously an adjacent pair and an anti-adjacent pair. The *endpoints* of the pair uv (regardless of adjacency) are u and v.

Note that each (non-empty, finite, and simple) graph can be thought of as a trigraph in a natural way: graphs are simply trigraphs with no semi-adjacent pairs.

The *complement* of a trigraph G is the trigraph $\overline{G}$ with vertex set $V_{\overline{G}} = V_G$ and adjacency function $\theta_{\overline{G}} = -\theta_G$. Given a non-empty set $X \subseteq V_G$, $G[X]$ is the trigraph with vertex set X and adjacency function $\theta_G \upharpoonright X \times X$ (the restriction of θ_G to $X \times X$); we call $G[X]$ the *subtrigraph of* G *induced by* X. Isomorphism between trigraphs is defined in the natural way; if trigraphs G_1 and G_2 are isomorphic, then we write $G_1 \cong G_2$. Given trigraphs G and H, we say that H is an *induced subtrigraph* of G (or that G *contains* H *as an induced subtrigraph*) if there exists some $X \subseteq V_G$ such that $H = G[X]$. (However, when convenient, we relax this condition and say that H is an induced subtrigraph of G, or that G contains H as an induced subtrigraph, if there exists some $X \subseteq V_G$ such that $H \cong G[X]$.) Given a trigraph G and a set $X \subseteq V_G$, we denote by $G \smallsetminus X$ the trigraph $G[V_G \smallsetminus X]$; if $X = \{v\}$, then we sometimes write $G \smallsetminus v$ instead of $G \smallsetminus \{v\}$.

A trigraph is called a *bull* provided that its vertex set is $\{x_1, x_2, x_3, y_1, y_2\}$, with adjacency as follows: $x_1x_2, x_2x_3, x_3x_1, x_1y_1, x_2y_2$ are adjacent pairs, and $x_1y_2, x_2y_1, x_3y_1, x_3y_2, y_1y_2$ are anti-adjacent pairs. A trigraph is said to be *bull-free* provided that no induced subtrigraph of it is a bull. An induced subtrigraph H of a trigraph G is a *hole* in G provided that the vertex set of H can be ordered as $\{h_1, ..., h_k\}$ (with $k \geq 4$), so that for all distinct $i, j \in \{1, ..., k\}$, if $|i-j| = 1$ or $|i-j| = k-1$ then h_ih_j is an adjacent pair, and if $1 < |i-j| < k-1$ then h_ih_j is an anti-adjacent pair. An induced subtrigraph H of a trigraph G is an *anti-hole* in G provided that $\overline{H}$ is a hole in $\overline{G}$. A hole or anti-hole is said to be *odd* if it has an odd number of vertices. A trigraph is said to be *Berge* if it contains neither an odd hole nor an odd anti-hole. We observe that every bull-free (respectively: Berge) graph can be seen as a bull-free (respectively: Berge) trigraph that has no semi-adjacent pairs.

A *clique* (respectively: *stable set*) in a trigraph is a set of pairwise adjacent (respectively: anti-adjacent) vertices; a *strong clique* (respectively: *strongly stable set*) is a set of pairwise strongly adjacent (respectively: strongly anti-adjacent) vertices. A trigraph is said to be *bipartite* if its vertex set can be partitioned into two (possibly empty) strongly stable sets.

A trigraph P is said to be a *path* provided that its vertex set can be ordered as $\{p_0, ..., p_k\}$ (where $k \geq 0$), so that for all distinct $i, j \in \{0, ..., k\}$, if $|i - j| = 1$ then $p_i p_j$ is an adjacent pair, and if $|i - j| > 1$ then $p_i p_j$ is an anti-adjacent pair. Under these circumstances, we say that p_0 and p_k are the *endpoints* of the path P. A *three-edge path* is a path that has exactly four vertices. A trigraph G is said to be *connected* provided that for all distinct $u, v \in V_G$, there exists an induced subtrigraph P of G such that P is a path with endpoints u and v.

Given a trigraph G, a vertex $a \in V_G$, and a set $B \subseteq V_G \smallsetminus \{a\}$, we say that a is *strongly complete* (respectively: *strongly anti-complete*, *complete*, *anti-complete*) to B provided that a is strongly adjacent (respectively: strongly anti-adjacent, adjacent, anti-adjacent) to every vertex in B; we say that a is *mixed* on B provided that a is neither strongly complete nor strongly anti-complete to B. Given disjoint sets $A, B \subseteq V_G$, we say that A is *strongly complete* (respectively: *strongly anti-complete*, *complete*, *anti-complete*) to B provided that for every $a \in A$, a is strongly complete (respectively: strongly anti-complete, complete, anti-complete) to B.

Let G be a trigraph, and let X be a proper, non-empty subset of V_G. We say that X is a *homogeneous set* in G provided that for every $v \in V_G \smallsetminus X$, v is either strongly complete to X or strongly anti-complete to X. A homogeneous set X in G is said to be *proper* provided that $|X| \geq 2$. We observe that if X is a homogeneous set in G, and uv is a semi-adjacent pair in G, then either $u, v \in X$ or $u, v \in V_G \smallsetminus X$.

Next, let G be a trigraph, and let A and B be non-empty, disjoint subsets of V_G. We say that (A, B) is a *homogeneous pair* in G provided that no vertex in $V_G \smallsetminus (A \cup B)$ is mixed on A, and no vertex in $V_G \smallsetminus (A \cup B)$ is mixed on B. We observe that if (A, B) is a homogeneous pair in G, and uv is a semi-adjacent pair in G, then either $u, v \in A \cup B$ or $u, v \in V_G \smallsetminus (A \cup B)$. If (A, B) is a homogeneous pair in G, and C is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are strongly complete to A and strongly anti-complete to B, D is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are strongly complete to B and strongly anti-complete to A, E is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are strongly complete to $A \cup B$, and F is the set of all vertices in $V_G \smallsetminus (A \cup B)$ that are strongly anti-complete to $A \cup B$, then we say that (A, B, C, D, E, F) is the *partition* of G associated with the homogeneous pair (A, B). A homogeneous pair (A, B) in G is said to be *reducible* provided that the associated partition (A, B, C, D, E, F) of G satisfies the following:

- either
  - $|B| \geq 3$, or
  - $|B| = 2$ and there exist distinct vertices $a, a' \in A$ such that a and a' are both mixed on B;
- C and D are both non-empty.

We observe that if (A, B) is a reducible homogeneous pair, then $|A \cup B| \geq 4$. We also observe that if (A, B) is a reducible homogeneous pair in a trigraph G, then (A, B) is a reducible homogeneous pair in $\overline{G}$ as well.

4 Transitively Orientable Trigraphs

A *directed trigraph* is an ordered pair $\vec{G} = (G, A_G)$, where $G = (V_G, \theta_G)$ is a trigraph, and $A_G \subseteq V_G \times V_G$ satisfies the following:

- for all $u \in V_G$, $(u, u) \notin A_G$;
- for all adjacent pairs uv in G, exactly one of (u, v) and (v, u) is in A_G;
- for all strongly anti-adjacent pairs uv in G, $(u, v) \notin A_G$.

Under these conditions, the directed trigraph $\vec{G}$ is said to be an *orientation* of the trigraph G; the set A_G is called the *arc set* of $\vec{G}$ and an *orientation relation* for G. Members of A_G are called the *arcs* of $\vec{G}$. An arc (u, v) in $\vec{G}$ is said to be *strong* provided that uv is a strongly adjacent pair in G. As in the case of (undirected) graphs, we note that every directed graph can be thought of as a directed trigraph in a natural way: a directed graph is simply a directed trigraph in which all arcs are strong.

A directed trigraph $\vec{G} = (G, A_G)$ is *transitive* provided that for all $u, v, w \in V_G$, if (u, v) and (v, w) are arcs in $\vec{G}$, then (u, w) is a strong arc in $\vec{G}$; under these circumstances, we say that A_G is a *transitive orientation relation* for G. A trigraph G is said to be *transitively orientable* provided that there exists a transitive orientation relation for it. As directed graphs are simply directed trigraphs in which all arcs are strong, it is easy to see that directed graph $\vec{G}$ is transitive as a graph if and only if it is transitive as a trigraph;

furthermore, every graph G is transitively orientable as a graph if and only if it is transitively orientable as a trigraph.

Our goal for the remainder of this section is to prove two theorems: one about "appropriate expansions" (defined below) of transitively orientable trigraphs (4.1), and one about trigraphs that contain no three-edge path as an induced subtrigraph (4.3). These two results will be used in the proofs of the decomposition theorem for bull-free Berge trigraphs (5.1), and in the proof of a certain result about reducible homogeneous pairs (5.2) in section 5.

We begin with "appropriate expansions." Given a trigraph G and a semi-adjacent pair ab in G, we say that ab is *expandable* provided that there exist vertices $c, d \in V_G \setminus \{a, b\}$ such that c is strongly adjacent to a and strongly anti-adjacent to b, and d is strongly adjacent to b and strongly anti-adjacent to a. A semi-adjacent pair ab in G is said to be *non-expandable* provided that it is not expandable. Given trigraphs H and G, we say that G is an *appropriate expansion* of H provided that there exists a family $\{X_v\}_{v \in V_H}$ of pairwise disjoint non-empty sets such that $V_G = \bigcup_{v \in V_H} X_v$, with all of the following satisfied:

- for all $v \in V_H$, if v is not an endpoint of any expandable semi-adjacent pair in H, then $|X_v| = 1$;
- if uv is a non-expandable semi-adjacent pair in H, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v in G;
- for all strongly adjacent pairs uv in H, X_u is strongly complete to X_v;
- for all strongly anti-adjacent pairs uv in H, X_u is strongly anti-adjacent to X_v.

We note that if a trigraph G is an appropriate expansion of a trigraph H, then $\overline{G}$ is an appropriate expansion of the trigraph $\overline{H}$.

4.1. *Let H be a transitively orientable trigraph, and let G be an appropriate expansion of H. Then at least one of the following holds:*

- *G is transitively orientable;*
- *G contains a proper homogeneous set;*
- *G contains a reducible homogeneous pair.*

Proof. Suppose first that $|X_v| \geq 3$ for some $v \in V_G$. Since $|X_v| > 1$, there exists some $u \in V_H$ such that uv is an expandable semi-adjacent pair in H. Let $(\{u\}, \{v\}, C, D, E, F)$ be the partition of H associated with the homogeneous pair $(\{u\}, \{v\})$; as uv is expandable, C and D are non-empty. But now (X_u, X_v) is a homogeneous pair in G with the associated partition

$(X_u, X_v, \bigcup_{w \in C} X_w, \bigcup_{w \in D} X_w, \bigcup_{w \in E} X_w, \bigcup_{w \in F} X_w)$; as C and D are non-empty, we know that $\bigcup_{w \in C} X_w$ and $\bigcup_{w \in D} X_w$ are both non-empty. Since $|X_v| \geq 3$, it follows that (X_u, X_v) is a reducible homogeneous pair in G, and we are done.

From now on, we assume that $|X_v| \leq 2$ for all $v \in V_H$. First, suppose that for some $v \in V_H$, we have that $|X_v| = 2$, and that more than one vertex in $V_G \smallsetminus X_v$ is mixed on X_v. As $|X_v| > 1$, we know that there exists some $u \in V_H$ such that uv is an expandable semi-adjacent pair in H. By the definition of an appropriate expansion, every vertex in $V_G \smallsetminus X_v$ that is mixed on X_v is a member of X_u; as more than one vertex of $V_G \smallsetminus X_v$ is mixed on X_v, this implies that at least two distinct vertices of X_u are mixed on X_v. Now, let $(\{u\}, \{v\}, C, D, E, F)$ be the partition of H associated with $(\{u\}, \{v\})$; as uv is expandable, we know that C and D are non-empty. But now (X_u, X_v) is a homogeneous pair in G with the associated partition $(X_u, X_v, \bigcup_{w \in C} X_w, \bigcup_{w \in D} X_w, \bigcup_{w \in E} X_w, \bigcup_{w \in F} X_w)$, and as C and D are non-empty, we know that $\bigcup_{w \in C} X_w$ and $\bigcup_{w \in D} X_w$ are non-empty. It follows that (X_u, X_v) is a reducible homogeneous pair in G, and we are done.

From now on, we assume that for all $v \in V_H$ such that $|X_v| = 2$, at most one vertex in $V_G \smallsetminus X_v$ is mixed on X_v. If there exists some $v \in V_H$ such that $|X_v| = 2$ and no vertex in $V_G \smallsetminus X_v$ is mixed on X_v, then X_v is a proper homogeneous set in G, and we are done. So we may assume that for all $v \in V_H$ such that $|X_v| = 2$, exactly one vertex in $V_G \smallsetminus X_v$ is mixed on X_v. Our goal now is to show that G is transitively orientable.

Since H is transitively orientable, there exists a transitive orientation relation A_H for the trigraph H; set $\vec{H} = (H, A_H)$, so that $\vec{H}$ is a transitive directed trigraph. Now, we define an orientation relation A_G for the trigraph G as follows. For all distinct $u, v \in V_H$ such that $(u, v) \notin A_H$, set $A_{u,v} = \emptyset$. For all distinct $u, v \in V_H$ such that $(u, v) \in A_H$, set $A_{u,v} = \{(\hat{u}, \hat{v}) \mid \hat{u} \in X_u, \hat{v} \in X_v, \theta_G(\hat{u}, \hat{v}) \geq 0\}$. For all $v \in V_H$, if X_v is a strongly stable set, set $A_{v,v} = \emptyset$. Now, suppose that $v \in V_H$ is such that $|X_v| = 2$, say $X_v = \{\hat{v}_1, \hat{v}_2\}$, where $\hat{v}_1\hat{v}_2$ is an adjacent pair. Then there exists a unique vertex $\hat{u} \in V_G \smallsetminus X_v$ such that $\hat{u}$ is mixed on X_v; fix distinct $i, j \in \{1, 2\}$ such that $\hat{u}$ is adjacent to v_i and anti-adjacent to v_j. Next, fix $u \in V_H$ such that $\hat{u} \in X_u$; then uv is an expandable semi-adjacent pair, and in particular, either $(u, v) \in A_H$ or $(v, u) \in A_H$. If $(u, v) \in A_H$, then set $A_{v,v} = \{(v_j, v_i)\}$; and if $(v, u) \in A_H$, then set $A_{v,v} = \{(v_i, v_j)\}$. Finally, set $A_G = \bigcup_{u,v \in V_G} A_{u,v}$ and $\vec{G} = (G, A_G)$. Clearly, $\vec{G}$ is a directed trigraph; we claim that $\vec{G}$ is a transitive.

Let $x, y, z \in V_G$, and assume that (x, y) and (y, z) are arcs in $\vec{G}$; we need to show that (x, z) is a strong arc in $\vec{G}$. First, we claim that there does not exist a vertex $v \in V_H$ such that $x, z \in X_v$. Suppose otherwise. Fix $v \in V_H$ such that $x, z \in X_v$; since $|X_v| \leq 2$, this implies that $X_v = \{x, z\}$. Next, fix $u \in V_H \setminus \{v\}$ such that $y \in X_u$. Now, since $(x, y) \in A_G$, $x \in X_v$, and $y \in X_u$, we know that $(v, u) \in V_H$; on the other hand, since $(y, z) \in A_G$, $y \in X_u$, and $z \in X_v$, we know that $(u, v) \in A_H$. But then (u, v) and (v, u) are both in A_H, which is impossible. This proves our claim.

Fix distinct $u, v \in V_H$ such that $x \in X_u$ and $z \in X_v$. There are three possibilities: that $y \in X_u$; that $y \in X_v$; and that $y \in X_w$ for some $w \in V_H \setminus \{u, v\}$. We note, however, that the cases when $y \in X_u$ and when $y \in X_v$ are very similar, and so it suffices to consider only the following two cases: when $y \in X_u$, and when $y \in X_w$ for some $w \in V_H \setminus \{u, v\}$.

Suppose first that $y \in X_u$. Since $(y, z) \in A_G$, $y \in X_u$, and $z \in X_v$, it follows that $(u, v) \in A_H$. Since $x \in X_u$ and $z \in X_v$, this implies that if xz is an adjacent pair in G then $(x, z) \in A_G$. Thus, it suffices to show that xz is a strongly adjacent pair in G. Since $(u, v) \in A_H$, we know that uv is an adjacent pair in G. If uv is a strongly adjacent pair in H, then since $x \in X_u$ and $z \in X_v$, we have that xz is a strongly adjacent pair, and we are done. So assume that uv is a semi-adjacent pair in H. Now, suppose that xz is not a strongly adjacent pair in G; then xz is an anti-adjacent pair. Since $(y, z) \in A_G$, we know that yz is an adjacent pair in G. Now yz is adjacent pair in G, xz is an anti-adjacent pair in G, and $x, y \in X_u$, $z \in X_v$, and $(u, v) \in A_H$; by construction then, $(y, x) \in A_G$. But this is impossible since $(x, y) \in A_G$. Thus, xz is a strongly adjacent pair, and we are done.

Suppose now that $y \in X_w$ for some $w \in V_H \setminus \{u, v\}$. Since $(x, y) \in A_G$, $x \in X_u$, and $y \in X_w$, we get that $(u, w) \in A_H$. Similarly, since $(y, z) \in A_G$, $y \in X_w$, and $z \in X_v$, we know that $(w, v) \in A_H$. Now since $(u, w), (w, v) \in A_H$, and $\vec{H}$ is transitive, we know that uv is a strongly adjacent pair in H and that $(u, v) \in A_H$. Since $x \in X_u$ and $z \in X_v$, it follows that xz is a strongly adjacent pair, and that $(x, z) \in A_G$. This completes the argument. □

Our goal for the remainder of this section is to prove that every trigraph that contains no three-edge path as an induced subtrigraph is transitively orientable (see 4.3 below). The proof of this result uses 4.5 from [6], which we state below.

4.2 (Chudnovsky [6]). *Let G be a trigraph that contains at least two vertices and that does not contain a three-edge path as an induced subtrigraph. Then at least one of the following holds:*

- *G is not connected;*
- *$\overline{G}$ is not connected;*

- *there exist vertices $x, y \in V_G$ such that x is semi-adjacent to y, x is strongly anti-complete to $V_G \smallsetminus \{x, y\}$, and y is strongly complete to $V_G \smallsetminus \{x, y\}$.*

We can now prove the final result of this section.

4.3. *Let G be a trigraph that does not contain a three-edge path as an induced subtrigraph. Then G is transitively orientable.*

Proof. We may assume inductively that for all non-empty sets $X \subsetneq V_G$, $G[X]$ is transitively orientable. Clearly, if $|V_G| \leq 2$, then G is transitively orientable; so assume that $|V_G| \geq 3$. By 4.2, G satisfies at least one of the following:

(i) G is not connected;

(ii) $\overline{G}$ is not connected;

(iii) there exist vertices $x, y \in V_G$ such that x is semi-adjacent to y, x is strongly anti-complete to $V_G \smallsetminus \{x, y\}$, and y is strongly complete to $V_G \smallsetminus \{x, y\}$.

Suppose first that (i) or (ii) holds, that is, that one of G and $\overline{G}$ is not connected. Fix disjoint, non-empty sets $X, Y \subseteq V_G$ such that $V_G = X \cup Y$ and such that X is either strongly complete or strongly anti-complete to Y. By assumption, $G[X]$ and $G[Y]$ are transitively orientable. Let $A_{G[X]}$ and $A_{G[Y]}$ be transitive orientation relations for $G[X]$ and $G[Y]$, respectively. If X is anti-complete to Y, then it is easy to see that $A_{G[X]} \cup A_{G[Y]}$ is a transitive orientation relation for G, and we are done. So assume that X is strongly complete to Y. Let $A'_G = \{(x, y) \mid x \in X, y \in Y\}$, and set $A_G = A_{G[X]} \cup A_{G[Y]} \cup A'_G$. We claim that A_G is a transitive orientation relation for G. Fix $u, v, w \in V_G$, and assume that $(u, v), (v, w) \in A_G$; we need to show that uw is a strongly adjacent pair in G and that $(u, w) \in A_G$. If $u \in X$ and $w \in Y$, then this is immediate. So assume that either $u \in Y$ or that $w \in X$. Suppose first that $u \in Y$. Since $u \in Y$ and $(u, v) \in A_G$, it follows that $v \notin X$, and consequently, $v \in Y$. Similarly, since $v \in Y$ and $(v, w) \in A_G$, it follows that $w \notin X$, and so $w \in Y$. But now $u, v, w \in Y$, and the result follows from the fact that $A_{G[Y]}$ is a transitive orientation relation for $G[Y]$. In a similar way, we get that if $w \in X$, then $u, v, w \in X$, and then the result follows from the fact that $A_{G[X]}$ is a transitive orientation relation for $G[X]$.

It remains to consider the case when (iii) holds. Set $Y = V_G \smallsetminus \{x, y\}$; then xy is a semi-adjacent pair, x is strongly anti-complete to Y, and y is strongly complete to Y. By assumption, $G \smallsetminus y$ is transitively orientable. Let $A_{G \smallsetminus y}$ be a transitive orientation relation for $G \smallsetminus y$. Let $A_y = \{(y, y') \mid y' \in Y\}$, and set $A_G = \{(y, x)\} \cup A_y \cup A_{G \smallsetminus y}$. We claim that A_G is a transitive orientation relation on G. Fix $u, v, w \in V_G$, and assume that $(u, v), (v, w) \in A_G$; we need to show that uw is a strongly adjacent pair, and that $(u, w) \in A_G$. If $y \notin \{u, v, w\}$, then the result follows from the fact that $A_{G \setminus y}$ is a transitive orientation relation for $G \setminus y$. So assume that $y \in \{u, v, w\}$. By construction, we have that for all $z \in V_G \setminus \{y\}$, $(z, y) \notin A_G$; thus, $y \neq v$ and $y \neq w$, and consequently, $y = u$. Note that x has only one neighbor in G (namely y), while v has two distinct neighbors in G (namely u and w); thus, $v \neq x$, and consequently, $v \in Y$. Since x is anti-complete to Y, and w is adjacent to $v \in Y$, it follows that $w \in Y$. But now $u = y$ and $w \in Y$, and so by construction, uw is a strongly adjacent pair in G, and $(u, w) \in A_G$. This completes the argument. $\square$

5 A Decomposition Theorem for Bull-Free Berge Trigraphs

The main goal of this section is to use the results of [8] and the results of section 4 above, to prove a decomposition theorem for bull-free Berge trigraphs (5.1), as well as a certain result about reducible homogeneous pairs (5.2), both of which we state below.

5.1. *Let G be a bull-free Berge trigraph. Then at least one of the following holds:*

- *G or $\overline{G}$ is transitively orientable;*
- *G contains a proper homogeneous set;*
- *G contains a reducible homogeneous pair.*

5.2. *Let G be a bull-free Berge trigraph that does not contain a proper homogeneous set, and let (A, B) be a reducible homogeneous pair in G. Then $G[A]$ and $G[B]$ are both transitively orientable.*

As every perfect graph is Berge, 2.1 and 2.2 (stated in section 2 above) are simply special cases of 5.1 and 5.2, respectively. We now turn to proving 5.1 and 5.2. Our first goal is to prove 5.1.

We begin with a definition, and then we state the structure theorem for bull-free Berge trigraphs from [8]. Let G, G_1, and G_2 be trigraphs, and let $v \in V_{G_1}$. Assume that V_{G_1} and V_{G_2} are disjoint, and that v is not an endpoint of any semi-adjacent pair in G_1. We then say that G is obtained by *substituting G_2 for v in G_1* provided that all of the following hold:

- $V_G = (V_{G_1} \setminus \{v\}) \cup V_{G_2}$;
- $G[V_{G_1} \setminus \{v\}] = G_1 \setminus v$;
- $G[V_{G_2}] = G_2$;

- for all $v_1 \in V_{G_1} \setminus \{v\}$ and $v_2 \in V_{G_2}$, v_1v_2 is a strongly adjacent pair in G if v_1v is a strongly adjacent pair in G_1, and v_1v_2 is a strongly anti-adjacent pair in G if v_1v is a strongly anti-adjacent pair in G_1.

We say that a trigraph G is obtained by *substitution from smaller trigraphs* provided that there exist some trigraphs G_1 and G_2 with disjoint vertex sets satisfying $|V_{G_1}| < |V_G|$ and $|V_{G_2}| < |V_G|$ (or equivalently: $|V_{G_1}| \geq 2$ and $|V_{G_2}| \geq 2$) and some vertex $v \in V_{G_1}$ that is not an endpoint of any semi-adjacent pair in G_1, such that G is obtained by substituting G_2 for v in G_1. We observe that if a trigraph G can be obtained by substitution from smaller trigraphs, then G contains a proper homogeneous set. The next theorem that we state is the structure theorem 3.4 for bull-free Berge trigraphs from [8]. (We note that we have not defined all the terms that appear in this theorem. In particular, we have not given the definitions of the classes $\mathcal{T}_1^*$ and $\mathcal{T}_2$, and we have not given the definition of an elementary expansion of a trigraph; all of these can be found in [8].)

5.3 (Chudnovsky and Penev [8]). *Let G be a trigraph. Then G is a bull-free Berge trigraph if and only if at least one of the following holds:*

- *G is obtained from smaller bull-free Berge trigraphs by substitution;*
- *G or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$;*
- *G is an elementary expansion of a trigraph in $\mathcal{T}_2$.*

Our goal is to use 5.3 to prove 5.1. First, we need a definition. A homogeneous pair (A, B) in a trigraph G is said to be *doubly dominating* provided that there exist non-empty sets $C, D \subseteq V_G$ such that $(A, B, C, D, \emptyset, \emptyset)$ is the partition of G associated with (A, B). It easily follows from the appropriate definitions (see [8]) that every trigraph that is an elementary expansion of a trigraph in $\mathcal{T}_2$ contains a doubly dominating homogeneous pair. We now use this fact to prove the theorem that follows.

5.4. *Let G be an elementary expansion of a trigraph in $\mathcal{T}_2$. Then at least one of the following holds:*

- *G is transitively orientable;*
- *G contains a proper homogeneous set;*
- *G contains a reducible homogeneous pair.*

Proof. Since G is an elementary expansion of a trigraph in $\mathcal{T}_2$, we know that G contains a doubly dominating homogeneous pair (A, B). Let $(A, B, C, D, \emptyset, \emptyset)$ be the partition of G associated with the homogeneous pair (A, B). Let H be a trigraph with vertex set $\{a, b, c, d\}$, where ab and cd are semi-adjacent pairs, ac and bd are strongly adjacent pairs, and ad and bc are strongly anti-adjacent pairs. Set $X_a = A$, $X_b = B$, $X_c = C$, and $X_d = D$.

With this set-up, it is easy to see that G is an appropriate expansion of H. Now, H is easily seen to be transitively orientable: for instance, $A_H = \{(a,b),(a,c),(d,b),(d,c)\}$ is a transitive orientation relation for H. But now by 4.1, we know that either G is transitively orientable, or G contains a proper homogeneous set, or G contains a reducible homogeneous pair. This completes the argument. □

We now turn to the elementary expansions of trigraphs in the class $\mathcal{T}_1^*$. Our goal is to prove 5.7 (stated later in this section), which is an analogue of 5.4 for trigraphs that are elementary expansions of trigraphs in the class $\mathcal{T}_1^*$. The definitions of an elementary expansion of a trigraph and of the class $\mathcal{T}_1^*$ are long and complicated, and we do not give them here (we refer the reader to [8]). Instead, we only state those properties of elementary expansions and of the class $\mathcal{T}_1^*$ that we need in order to prove 5.7. First, it follows immediately from the definition of an elementary expansion of a trigraph (see [8]) that if a trigraph G is a elementary expansion of a trigraph H, then G is an appropriate expansion of H; we state this result below for future reference.

5.5. *Let G be an elementary expansion of a trigraph H. Then G is an appropriate expansion of H.*

In view of 4.1 and 5.5, we note that in order to prove 5.7, it suffices to show that every trigraph in the class $\mathcal{T}_1^*$ is transitively orientable. We start with a couple of definitions. Let G be a trigraph such that $V_G = K \cup A \cup B$, where K, A, and B are pairwise disjoint. Assume that $K = \{k_1, ..., k_t\}$ is a strong clique, and that A and B are strongly stable sets. Let $A = \bigcup_{i=1}^t A_i$ and $B = \bigcup_{i=1}^t B_i$, and assume that $A_1, ..., A_t$, $B_1, ..., B_t$ are pairwise disjoint. Assume that for all $i \in \{1, ..., t\}$, the following hold:

- A_i is strongly complete to $\{k_1, ..., k_{i-1}\}$;
- A_i is complete to $\{k_i\}$;
- A_i is strongly anti-complete to $\{k_{i+1}, ..., k_t\}$;
- B_i is strongly complete to $\{k_{t-i+2}, ..., k_t\}$;
- B_i is complete to $\{k_{t-i+1}\}$;
- B_i is strongly anti-complete to $\{k_1, ..., k_{t-i}\}$.

For each $i \in \{1, ..., t\}$, let A_i' be the set of all vertices in A_i that are semi-adjacent to k_i, and let B_i' be the set of all vertices in B_i that are semi-adjacent to k_{t-i+1} (thus, $|A_i'| \leq 1$ and $|B_i'| \leq 1$). Next, assume that:

- for all $i, j \in \{1, ..., t\}$ such that $i + j \neq t$, A_i is strongly complete to B_j;

- for all $i \in \{1, ..., t\}$, A'_i is strongly complete to B_{t-i}, B'_{t-i} is strongly complete to A_i, and the adjacency between $A_i \setminus A'_i$ and $B_{t-i} \setminus B'_{t-i}$ is arbitrary;
- A_t and B_t are both non-empty.

We then say that G is a (K, A, B)-*tulip*.

We say that a trigraph G is a *tulip bed* provided that either G is bipartite, or V_G can be partitioned into (non-empty) sets $F_1, F_2, Y_1, ..., Y_s$ (for some integer $s \geq 1$) such that all of the following hold:

- F_1 and F_2 are strongly stable sets;
- $Y_1, ..., Y_s$ are strong cliques, pairwise strongly anti-complete to each other;
- for all $v \in F_1 \cup F_2$, v has neighbors in at most two of $Y_1, ..., Y_s$;
- for all adjacent $v_1 \in F_1$ and $v_2 \in F_2$, v_1 and v_2 have common neighbors in at most one of $Y_1, ..., Y_s$;
- for all $l \in \{1, ..., s\}$, if X_l is the set of all vertices in $F_1 \cup F_2$ with a neighbor in Y_l, then $G[Y_l \cup X_l]$ is a $(Y_l, X_l \cap F_1, X_l \cap F_2)$-tulip.

5.8 from [8] states that every trigraph in $\mathcal{T}_1^*$ is a tulip bed (we note, however, that not every tulip bed is in $\mathcal{T}_1^*$; indeed, not every tulip bed is bull-free). We now prove that every tulip bed (and consequently, every trigraph in $\mathcal{T}_1^*$) is transitively orientable.

5.6. *Every tulip bed is transitively orientable. Consequently, every trigraph in $\mathcal{T}_1^*$ is transitively orientable.*

Proof. Since every trigraph in $\mathcal{T}_1^*$ is a tulip bed, it suffices to show that every tulip bed is transitively orientable. Let G be a tulip bed. Suppose first that G is bipartite, and let A and B be disjoint strongly stable sets in G such that $V_G = A \cup B$. Now, set $A_G = \{(a, b) \mid a \in A, b \in B, \theta_G(a, b) \geq 0\}$. Clearly, A_G is a transitive orientation relation for G. So from now on, we assume that G is not bipartite. Then let $F_1, F_2, Y_1, ..., Y_s, X_1, ..., X_s$ be as in the definition of a tulip bed. It suffices to show that for all $l \in \{1, ..., s\}$, there exists a transitive orientation relation A_l for $G[Y_l \cup X_l]$ such that all of the following hold:

- for all adjacent $x_1 \in X_l \cap F_1$ and $x_2 \in X_l \cap F_2$, we have that $(x_1, x_2) \in A_l$;
- for all adjacent $x_1 \in X_l \cap F_1$ and $y \in Y_l$, we have that $(x_1, y) \in A_l$;
- for all adjacent $x_2 \in X_l \cap F_2$ and $y \in Y_l$, we have that $(y, x_2) \in A_l$.

Indeed, if such transitive orientation relations $A_1, ..., A_s$ exist, then the fact that F_1 and F_2 are strongly stable, and the fact that the cliques $Y_1, ..., Y_s$ are pairwise strongly anti-complete to each other, will immediately imply that $A_G = \{(v_1, v_2) \mid v_1 \in F_1, v_2 \in F_2, \theta_G(v_1, v_2) \geq 0\} \cup \bigcup_{l=1}^{s} A_l$ is a transitive orientation relation for G.

Fix $l \in \{1, ..., s\}$, and set $K = Y_l$, $A = X_l \cap F_1$, and $B = X_l \cap F_2$; then $G[Y_l \cup X_l]$ is a (K, A, B)-tulip. Set $K = \{k_1, ..., k_t\}$ as in the definition of a (K, A, B)-tulip. Then we let A_l be the union of the following four sets:

$$\begin{array}{l} \{(a, b) \mid a \in A, b \in B, \theta_G(a, b) \geq 0\}; \\ \{(a, k) \mid a \in A, k \in K, \theta_G(a, k) \geq 0\}; \\ \{(k, b) \mid k \in K, b \in B, \theta_G(k, b) \geq 0\}; \\ \{(k_j, k_i) \mid i, j \in \{1, ..., t\}, i < j\}. \end{array}$$

Then A_l is easily seen to be a transitive orientation relation on $G[Y_l \cup X_l]$, satisfying the three requirements described above. This completes the argument. □

We can now prove an analogue of 5.4 for trigraphs that are elementary expansions of trigraphs in the class $\mathcal{T}_1^*$.

5.7. *Let G be an elementary expansion of a trigraph in the class $\mathcal{T}_1^*$. Then at least one of the following holds:*

- *G is transitively orientable;*
- *G contains a proper homogeneous set;*
- *G contains a reducible homogeneous pair.*

Proof. By 5.5, G is an appropriate expansion of a trigraph in the class $\mathcal{T}_1^*$, and by 5.6, every trigraph in $\mathcal{T}_1^*$ is transitively orientable. But now the result follows from 4.1. □

Finally, we restate and prove the main result of this section, a decomposition theorem for bull-free Berge trigraphs.

5.1. *Let G be a bull-free Berge trigraph. Then at least one of the following holds:*

- *G or $\overline{G}$ is transitively orientable;*
- *G contains a proper homogeneous set;*
- *G contains a reducible homogeneous pair.*

Proof. By 5.3, we know that at least one of the following holds:

- G is obtained from smaller bull-free Berge trigraphs by substitution;

- G or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$;
- G is an elementary expansion of a trigraph in $\mathcal{T}_2$.

If G can be obtained from smaller bull-free Berge trigraphs by substitution, then G has a proper homogeneous set, and we are done. If G is an elementary expansion of a trigraph in $\mathcal{T}_1^* \cup \mathcal{T}_2$, then the result follows from 5.4 and 5.7. It remains to consider the case when $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$. Then by 5.7, at least one of the following holds:

- $\overline{G}$ is transitively orientable;
- $\overline{G}$ contains a proper homogeneous set;
- $\overline{G}$ contains a reducible homogeneous pair.

If $\overline{G}$ is transitively orientable, then we are done. And if $\overline{G}$ contains a proper homogeneous set or a reducible homogeneous pair, then so does G, and again we are done. □

We now turn to the proof of 5.2, stated at the beginning of this section. We first need some definitions. Given a trigraph G, an induced subtrigraph H of G, and a vertex $v \in V_G \setminus V_H$, we say that v is a *center* (respectively: an *anti-center*) for H provided that v is complete (respectively: anti-complete) to V_H. We say that a bull-free trigraph G is *non-elementary* provided that there exists an induced subtrigraph P of G, and distinct vertices $c, a \in V_G \setminus V_P$, such that P is a three-edge path, c is a center for P, and a is an anti-center for P. A bull-free trigraph G is said to be *elementary* provided that it is not non-elementary. The following result is 3.2 from [8].

5.8 (Chudnovsky and Penev [8])**.** *Let G be a non-elementary bull-free Berge trigraph. Then G contains a proper homogeneous set.*

We now restate and prove 5.2.

5.2. *Let G be a bull-free Berge trigraph that does not contain a proper homogeneous set, and let (A, B) be a reducible homogeneous pair in G. Then $G[A]$ and $G[B]$ are both transitively orientable.*

Proof. First, by 5.8, G is elementary. Next, by 4.3, every trigraph that does not contain a three-edge path as an induced subtrigraph is transitively orientable. Thus, it suffices to show that neither $G[A]$ nor $G[B]$ contains a three-edge path as an induced subtrigraph. Let (A, B, C, D, E, F) be the partition of G associated with (A, B). Since (A, B) is reducible, we know that C and D are both non-empty; fix $c \in C$ and $d \in D$. Now c is a center and d is an anti-center for $G[A]$, and so since G is elementary, $G[A]$ does not contain a three-edge path as an induced subtrigraph. Similarly, d is a center and c is an anti-center for $G[B]$, and so $G[B]$ does not contain a three-edge path as an induced subtrigraph. This completes the argument. □

6 Reducible Homogeneous Pairs

Our main goal in this section is to describe the algorithm REDUCIBLE, which, given a graph G that does not contain a proper homogeneous set, either finds a reducible homogeneous pair in G, or determines that G does not contain such a homogeneous pair. We begin with some definitions. Given a graph G, and a triple (b, b', d) of pairwise distinct vertices, we say that (b, b', d) is a *reducible frame* in G provided that there exists a reducible homogeneous pair (A, B) in G with associated partition (A, B, C, D, E, F) such that the following hold:

- $b, b' \in B$;
- $d \in D$;
- at least one of the following holds:
  - some vertex $b'' \in B \smallsetminus \{b, b'\}$ is mixed on $\{b, b'\}$,
  - there exist distinct vertices $a, a' \in A$ such that a and a' are both mixed on $\{b, b'\}$.

Under these circumstances, we also say that (b, b', d) is a *frame* for the reducible homogeneous pair (A, B). We observe that if (b, b', d) is reducible frame, then all of the following hold:

- d is complete to $\{b, b'\}$;
- at least one of the following holds:
  - there exists a vertex $b'' \in V_G \smallsetminus \{b, b', d\}$ such that b'' is mixed on $\{b, b'\}$ and adjacent to d,
  - there exist distinct vertices $a, a' \in V_G \smallsetminus \{b, b', d\}$ such that a and a' are both mixed on $\{b, b'\}$ and non-adjacent to d.

Reducible frames will be our main tool for detecting reducible homogeneous pairs in graphs that contain no proper homogeneous sets. But first, we need the following result.

6.1. *Let G be a graph that does not contain a proper homogeneous set, and let (A, B) be a reducible homogeneous pair in G. Then G contains a frame for (A, B).*

Proof. Let (A, B) be a reducible homogeneous pair in G, and let (A, B, C, D, E, F) be the associated partition of G. By the definition of a reducible homogeneous pair, we know that D is non-empty.

Suppose first that $|B| = 2$, say $B = \{b, b'\}$, and that there exist distinct vertices $a, a' \in A$ such that a and a' are both mixed on B. Now, using the fact that D is non-empty, we fix some $d \in D$, and observe that (b, b', d) is a frame for (A, B).

It remains to consider the case when $|B| \geq 3$. Suppose first that B is neither a clique nor a stable set. Then there exist vertices b, b', b'' such that b'' is adjacent to b and non-adjacent to b'. Fix some $d \in D$. Then (b, b', d) is a frame for (A, B).

Suppose now that B is either a clique or a stable set. Since B is not a homogeneous set in G, some vertex $a \in V_G \smallsetminus B$ is mixed on B; since (A, B) is a homogeneous pair, we know that $a \in A$. Let $N_1(a)$ and $N_2(a)$ be the sets of neighbors and non-neighbors, respectively, of a in B. Since a is mixed on B, both $N_1(a)$ and $N_j(a)$ are non-empty. Let $i, j \in \{1, 2\}$ be distinct with the property that $|N_i(a)| \geq |N_j(a)|$. Since B is the disjoint union of the sets $N_i(a)$ and $N_j(a)$, and since $|B| \geq 3$, we know that $|N_i(a)| \geq 2$. Since $N_i(a)$ is not a homogeneous set in G, some vertex $a' \in V_G \smallsetminus N_i(a)$ is mixed on $N_i(a)$. Now, a' is mixed on $N_i(a) \subseteq B$, B is either a clique or a stable set, and (A, B) is homogeneous pair in G; it follows that $a' \in A$. Fix $b_i, b'_i \in N_i(a)$ such that a' is adjacent to b_i and non-adjacent to b'_i, and fix some $b_j \in N_j(a)$. By construction, a is mixed on both $\{b_i, b_j\}$ and $\{b'_i, b_j\}$. Further, if a' is adjacent to b_j, then a' is mixed on $\{b'_i, b_j\}$, and if a' is non-adjacent to b_j, then a' is mixed on $\{b_i, b_j\}$. Thus, we have that either a and a' are both mixed on $\{b_i, b_j\}$, or a and a' are both mixed on $\{b'_i, b_j\}$. Now, fix some $d \in D$. Then at least one of (b_i, b_j, d) and (b'_i, b_j, d) is a frame for (A, B). This completes the argument. □

Given a graph G, a reducible homogeneous pair (A, B) in G, and a frame (b, b', d) for (A, B), we say that (A, B) is the *minimal* reducible homogeneous pair for the reducible frame (b, b', d) provided that for all reducible homogeneous pairs (A', B') in G such that (b, b', d) is a frame for (A', B'), we have that $A \subseteq A'$ and $B \subseteq B'$. Our next result (6.2) establishes that for every graph G, and every reducible frame (b, b', d) in G, there exists a unique minimal reducible homogeneous pair in G for (b, b', d). We note that the proof of 6.2 can easily be turned into an algorithm that, given a graph G that does not contain a proper homogeneous set, and a triple (b, b', d) of pairwise distinct vertices in G, either returns a 6-tuple (A, B, C, D, E, F) such that (A, B) is the unique minimal reducible homogeneous pair in G for the reducible frame (b, b', d), and the associated partition of G is (A, B, C, D, E, F), or determines that (b, b', d) is not a reducible frame in G. The running time of the algorithm is $O(n^2)$, where $n = |V_G|$.

6.2. *Let G be a graph that does not contain a proper homogeneous set, and let (b, b', d) be a triple of pairwise distinct vertices in G. Then if (b, b', d) is a reducible frame in G, then there exists a unique minimal reducible homogeneous pair (A, B) for (b, b', d).*

Proof. If d is not complete to $\{b, b'\}$, then (b, b', d) is not a reducible frame, and there is nothing to show. So assume that d is complete to $\{b, b'\}$. Next, we let S_A be the set of all vertices in $V_G \smallsetminus \{b, b', d\}$ that are mixed on $\{b, b'\}$ and non-adjacent to d, and we let S_B be the set of all vertices in $V_G \smallsetminus \{b, b', d\}$ that are mixed on $\{b, b'\}$ and adjacent to d. If $|S_A| \leq 1$ and $S_B = \emptyset$, then (b, b', d) is not a reducible frame, and again, there is nothing to show. So assume that either $|S_A| \geq 2$ or $S_B \neq \emptyset$. We note that if (A', B') is a reducible homogeneous pair in G such that (b, b', d) is a frame for (A', B'), then the fact that every vertex in $S_A \cup S_B$ is mixed on $\{b, b'\} \subseteq B'$ implies that $S_A \cup S_B \subseteq A' \cup B'$, and then the fact that d is complete to A' and anti-complete B' implies that $S_A \subseteq A'$ and $S_B \subseteq B'$.

Now, we construct sets A and B, as well as the function $l : V_G \smallsetminus (A \cup B \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}, \mathrm{M}\} \times \{\mathrm{C}, \mathrm{A}, \mathrm{M}\}$, as follows. (Note: "E" stands for "empty," "C" stands for "complete," "A" stands for "anti-complete," and "M" stands for "mixed.")

First, set $A_0 = S_A$ and $B_0 = \{b, b'\} \cup S_B$. (Note that A_0 may be empty, but B_0 is non-empty.) Next, define the function $l_0 : V_G \smallsetminus (A_0 \cup B_0 \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}, \mathrm{M}\} \times \{\mathrm{C}, \mathrm{A}, \mathrm{M}\}$ as follows. For all $v \in V_G \smallsetminus (A_0 \cup B_0 \cup \{d\})$, set $l_0(v) = (X, Y)$, where:

- if $A_0 = \emptyset$, then we set $X = \mathrm{E}$;
- if $A_0 \neq \emptyset$ and v is complete to A_0, then we set $X = \mathrm{C}$;
- if $A_0 \neq \emptyset$ and v is anti-complete to A_0, then we set $X = \mathrm{A}$;
- if $A_0 \neq \emptyset$ and v is mixed on A_0, then we set $X = \mathrm{M}$;
- if v is complete to B_0, then we set $Y = \mathrm{C}$;
- if v is anti-complete to B_0, then we set $Y = \mathrm{A}$;
- if v is mixed on B_0, then we set $Y = \mathrm{M}$.

Assume now that we have constructed sets A_i and B_i, as well as a function $l_i : V_G \smallsetminus (A_i \cup B_i \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}, \mathrm{M}\} \times \{\mathrm{C}, \mathrm{A}, \mathrm{M}\}$. If every vertex $u \in V_G \smallsetminus (A_i \cup B_i \cup \{d\})$ satisfies the property that $l_i(u) \in \{\mathrm{E}, \mathrm{C}, \mathrm{A}\} \times \{\mathrm{C}, \mathrm{A}\}$, then we terminate the construction, and we set $A = A_i$, $B = B_i$, and $l = l_i$. Suppose now that there exists some vertex $u \in V_G \smallsetminus (A_i \cup B_i \cup \{d\})$ such that at least one coordinate of $l_i(u)$ is M. In this case, we construct sets A_{i+1} and B_{i+1} as follows. If u is adjacent to d, then we set $A_{i+1} = A_i$ and $B_{i+1} = B_i \cup \{u\}$, and we define a function $l_{i+1} : V_G \smallsetminus (A_{i+1} \cup B_{i+1} \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}, \mathrm{M}\} \times \{\mathrm{C}, \mathrm{A}, \mathrm{M}\}$ in such a way that for all $v \in V_G \smallsetminus (A_{i+1} \cup B_{i+1} \cup \{d\})$ with $l_i(v) = (X, Y)$, we set $l_{i+1}(v) = (X', Y')$, where:

- $X' = X$;

- if $Y \in \{\text{C}, \text{M}\}$ and v is adjacent to u, then $Y' = Y$;
- if $Y = \text{A}$ and v is adjacent to u, then $Y' = \text{M}$;
- if $Y \in \{\text{A}, \text{M}\}$ and v is non-adjacent to u, then $Y' = Y$;
- if $Y = \text{C}$ and v is non-adjacent to u, then $Y' = \text{M}$.

On the other hand, if u is non-adjacent to d, then we set $A_{i+1} = A_i \cup \{u\}$ and $B_{i+1} = B_i$, and we define a function $l_{i+1} : V_G \smallsetminus (A_{i+1} \cup B_{i+1} \cup \{d\}) \to \{\text{E}, \text{C}, \text{A}, \text{M}\} \times \{\text{C}, \text{A}, \text{M}\}$ in such a way that for all $v \in V_G \smallsetminus (A_{i+1} \cup B_{i+1} \cup \{d\})$ with $l_i(v) = (X, Y)$, we set $l_{i+1}(v) = (X', Y')$, where:

- $Y' = Y$;
- if $X = \text{E}$ and v is adjacent to u, then $X' = \text{C}$;
- if $X = \text{E}$ and v is non-adjacent to u, then $X' = \text{A}$;
- if $X \in \{\text{C}, \text{M}\}$ and v is adjacent to u, then $X' = X$;
- if $X = \text{A}$ and v is adjacent to u, then $X' = \text{M}$;
- if $X \in \{\text{A}, \text{M}\}$ and v is non-adjacent to u, then $X' = X$;
- if $X = \text{C}$ and v is non-adjacent to u, then $X' = \text{M}$.

Now, we may assume that the construction above yields sequences of sets $A_0, ..., A_n$ and $B_0, ..., B_n$, as well as a sequence of functions $l_0 : V_G \smallsetminus (A_0 \cup B_0 \cup \{d\}) \to \{\text{E}, \text{C}, \text{A}, \text{M}\} \times \{\text{C}, \text{A}, \text{M}\}, ..., l_{n-1} : V_G \smallsetminus (A_{n-1} \cup B_{n-1} \cup \{d\}) \to \{\text{E}, \text{C}, \text{A}, \text{M}\} \times \{\text{C}, \text{A}, \text{M}\}, l_n : V_G \smallsetminus (A_n \cup B_n \cup \{d\}) \to \{\text{E}, \text{C}, \text{A}\} \times \{\text{C}, \text{A}\}$, such that $A = A_n$, $B = B_n$, and $l = l_n$. We claim that for all $i \in \{0, ..., n\}$, the following hold:

- $S_A \subseteq A_i$;
- $S_B \cup \{b, b'\} \subseteq B_i$;
- $d \notin A_i \cup B_i$;
- d is anti-complete to A_i and complete to B_i;
- for all $v \in V_G \smallsetminus (A_i \cup B_i \cup \{d\})$ with $l_i(v) = (X, Y)$, the following hold:
  - if $X = \text{E}$, then A_i is empty,
  - if $X = \text{C}$, then A_i is non-empty and v is complete to A_i,
  - if $X = \text{A}$, then A_i is non-empty and v is anti-complete to A_i,
  - if $X = \text{M}$, then A_i is non-empty and v is mixed on A_i,
  - if $Y = \text{C}$, then v is complete to B_i,
  - if $Y = \text{A}$, then v is anti-complete to B_i,

- if $Y = \mathrm{M}$, then v is mixed on B_i;

- for all reducible homogeneous pairs (A', B') such that (b, b', d) is a frame for (A', B'), we have that $A_i \subseteq A'$ and $B_i \subseteq B'$.

We prove this by induction on i. For the base case, this is immediate by construction. For the induction step, we assume that the claim holds for some $i \in \{0, ..., n-1\}$, and we show that it holds for $i+1$. All requirements except for the last one are easily seen to follow from the induction hypothesis and the construction. For the last requirement, suppose that (A', B') is a reducible homogeneous pair in G such that (b, b', d) is a frame for (A', B'). By the induction hypothesis, $A_i \subseteq A'$ and $B_i \subseteq B'$. Furthermore, since (b, b', d) is a frame for (A', B'), we know that $d \notin A' \cup B'$, and that d is complete to B' and anti-complete to A'. Now, fix $u \in V_G \setminus (A_i \cup B_i \cup \{d\})$ such that either $A_{i+1} = A_i$ and $B_{i+1} = B_i \cup \{u\}$, or $A_{i+1} = A_i \cup \{u\}$ and $B_{i+1} = B_i$. By construction, we know that at least one coordinate of $l_i(u)$ is M, and so by the induction hypothesis, u is mixed on at least one of A_i and B_i. Since $A_i \subseteq A'$ and $B_i \subseteq B'$, and since (A', B') is a homogeneous pair, it follows that $u \in A' \cup B'$. Since d is anti-complete to A' and complete to B', we have that if u is adjacent to d then $u \in B'$, and if u is non-adjacent to d then $u \in A'$. By construction then, we get that $A_{i+1} \subseteq A'$ and $B_{i+1} \subseteq B'$. This completes the induction. Now, by construction, we have that $A = A_n$, $B = B_n$, and $l = l_n$; furthermore, we know that $l_n(v) \in \{\mathrm{E}, \mathrm{C}, \mathrm{A}\} \times \{\mathrm{C}, \mathrm{A}\}$ for all $v \in V_G \setminus (A_n \cup B_n \cup \{d\})$. By what we just showed, this implies the following:

- $S_A \subseteq A$;
- $S_B \cup \{b, b'\} \subseteq B$;
- $d \notin A \cup B$;
- d is anti-complete to A and complete to B;
- $l : V_G \setminus (A \cup B \cup \{d\}) \to \{\mathrm{E}, \mathrm{C}, \mathrm{A}\} \times \{\mathrm{C}, \mathrm{A}\}$;
- for all $v \in V_G \setminus (A \cup B \cup \{d\})$ with $l(v) = (X, Y)$, the following hold:
  - if $X = \mathrm{E}$, then A is empty,
  - if $X = \mathrm{C}$, then A is non-empty and v is complete to A,
  - if $X = \mathrm{A}$, then A is non-empty and v is anti-complete to A,
  - if $Y = \mathrm{C}$, then v is complete to B,
  - if $Y = \mathrm{A}$, then v is anti-complete to B,
- for all reducible homogeneous pairs (A', B') such that (b, b', d) is a frame for (A', B'), we have that $A \subseteq A'$ and $B \subseteq B'$.

Note first that the above implies that no vertex in $V_G \setminus (A \cup B)$ is mixed on either A or B. Next, note that A is non-empty, for otherwise, B would be a proper homogeneous set in G, and by assumption, G has no proper homogeneous sets; clearly, this implies that for all $v \in V_G \setminus (A \cup B \cup \{d\})$, $l(v) \in \{\mathrm{C}, \mathrm{A}\} \times \{\mathrm{C}, \mathrm{A}\}$. Now, since $S_A \subseteq A$, $S_B \cup \{b, b'\} \subseteq B$, and either $|S_A| \geq 2$ or $S_B \neq \emptyset$, we get that either

- $|B| \geq 3$, or
- $|B| = 2$ and there exist distinct vertices $a, a' \in A$ such that a and a' are both mixed on B.

Next, let C be the set of all vertices $v \in V_G \setminus (A \cup B \cup \{d\})$ such that $l(v) = (\mathrm{C}, \mathrm{A})$; let D be the set consisting of the vertex d as well as of all the vertices $v \in V_G \setminus (A \cup B \cup \{d\})$ such that $l(v) = (\mathrm{A}, \mathrm{C})$; let E be the set of all vertices $v \in V_G \setminus (A \cup B \cup \{d\})$ such that $l(v) = (\mathrm{C}, \mathrm{C})$; and let F be the set of all vertices $v \in V_G \setminus (A \cup B \cup \{d\})$ such that $l(v) = (\mathrm{A}, \mathrm{A})$. Note that the fact that $d \in D$ implies that D is non-empty. It now easily follows that (A, B) is a homogeneous pair in G with associated partition (A, B, C, D, E, F). Now, we claim that if C is non-empty then (A, B) is the unique minimal reducible homogeneous pair for (b, b', d) in G, and if C is empty then (b, b', d) is not a reducible frame.

Suppose first that C is non-empty. Since $d \in D$, we know that D is non-empty. We showed above that either $|B| \geq 3$, or $|B| = 2$ and there exist distinct vertices $a, a' \in A$ such that a and a' are both mixed on B. Thus, (A, B) is a reducible homogeneous pair in G. The fact that(b, b', d) is a frame for (A, B) follows from the fact that either $|S_A| \geq 2$ or $S_B \neq \emptyset$. The minimality of (A, B) follows from the fact that for all reducible homogeneous pairs (A', B') such that (b, b', d) is a frame for (A', B'), we have that $A \subseteq A'$ and $B \subseteq B'$. The uniqueness of (A, B) is immediate.

Suppose now that C is empty. Suppose that (b, b', d) is a reducible frame, and fix a reducible homogeneous pair (A', B') in G such that (b, b', d) is a frame for (A', B'). Let (A', B', C', D', E', F') be the partition of G associated with (A', B'). We know that $A \subseteq A'$ and $B \subseteq B'$, and so we have that $C' \subseteq C$. But now since C is empty, so is C', contrary to the assumption that (A', B') is a reducible homogeneous pair. This completes the argument. □

We now prove an easy lemma, and then we turn to the algorithm REDUCIBLE.

6.3. *Let G be a graph that does not contain a proper homogeneous set, let (A, B) be a reducible homogeneous pair in G, and let (b, b', d) be a frame for (A, B). Then the following hold:*

- *if b is adjacent to b', then there exist vertices $a \in A$ and $b_1, b_2 \in B$ such that ab_1 and b_1b_2 are edges, and ab_2 is a non-edge;*

- *if b is non-adjacent to b', then there exist vertices $a \in A$ and $b_1, b_2 \in B$ such that ab_1 is an edge, and ab_2 and b_1b_2 are non-edges.*

Proof. If b is adjacent to b', then $G[B]$ contains a non-trivial component (i.e. a component that has at least two vertices), and if b is non-adjacent to b', then $\overline{G}[B]$ contains a non-trivial component. If bb' is an edge, let $W \subseteq B$ be such that $G[W]$ is a non-trivial component of $G[B]$; and if bb' is a non-edge, then let $W \subseteq B$ be such that $\overline{G}[W]$ is a non-trivial component of $\overline{G}[B]$. Since $|W| \geq 2$, and G contains no proper homogeneous set, we know that some vertex $a \in A$ is mixed on W. If bb' is an edge, so that $G[W]$ is a component of $G[B]$, then there exist adjacent vertices $b_1, b_2 \in W$ such that a is adjacent to b_1 and non-adjacent to b_2. And if bb' is a non-edge, so that $\overline{G}[W]$ is a component of $\overline{G}[B]$, then there exist non-adjacent vertices $b_1, b_2 \in W$ such that a is adjacent to b_1 and non-adjacent to b_2. This completes the argument. □

We now describe the algorithm REDUCIBLE that, given a graph G that does not contain a proper homogeneous set, either returns a 7-tuple (A, B, C, D, E, F, z) such that the following hold:

- (A, B) is a reducible homogeneous pair, and (A, B, C, D, E, F) is the associated partition of G;
- $z \in \{\text{a}, \text{n}\}$;
- if $z = \text{a}$, then there exist vertices $a \in A$ and $b_1, b_2 \in B$ such that ab_1 and b_1b_2 are edges, and ab_2 is a non-edge;
- if $z = \text{n}$, then there exist vertices $a \in A$ and $b_1, b_2 \in B$ such that ab_1 is an edge, and ab_2 and b_1b_2 are non-edges;

or determines that G does not contain a reducible homogeneous pair.

We enumerate all triples (b, b', d) of pairwise distinct vertices in G. For each such triple (b, b', d), we call the algorithm from 6.2, and either obtain a 6-tuple (A, B, C, D, E, F) such that (A, B) is a reducible homogeneous pair in G such that (b, b', d) is a frame for (A, B) and (A, B, C, D, E, F) is the associated partition of G, or we obtain the answer that (b, b', d) is not a reducible frame. If we obtain a 6-tuple (A, B, C, D, E, F), then using 6.3, if bb' is an edge then we set $z = \text{a}$, and if bb' is a non-edge then we set $z = \text{n}$; we then stop, and the algorithm returns the 7-tuple (A, B, C, D, E, F, z). If we obtained the answer that (b, b', d) is not a reducible frame, then we move to the next triple on the list, and repeat the process. If for every triple (b, b', d) on the list, the algorithm determines that (b, b', d) is not a reducible frame, then by 6.1, it follows that G does not contain a reducible homogeneous pair; in this case, we stop, and the algorithm returns the answer that G contains no reducible homogeneous pair.

We observe that the running time of the algorithm REDUCIBLE is at most $O(n^5)$, where n is the number of vertices of the input graph.

7 Reducing Homogeneous Sets and Reducible Homogeneous Pairs

In this section, we describe "reductions" of weighted graphs with respect to proper homogeneous sets and with respect to reducible homogeneous pairs. We also explain how to use these reductions to "recover" a maximum weighted clique in the original graph.

We first deal with reductions with respect to proper homogeneous sets. Suppose that G is a weighted graph, and that S is a proper homogeneous set in G. Let $\tilde{G}$ be the graph whose vertex set is $(V_G \smallsetminus S) \cup \{s\}$, where $s \notin V_G$, with weights assigned as follows: $w_{\tilde{G}}(v) = w_G(v)$ for all $v \in V_G \smallsetminus S$; and $w_{\tilde{G}}(s) = W(G[S])$. We refer to the weighted graph $\tilde{G}$ as the *reduction of G with respect to S*. Note that, as an unweighted graph, $\tilde{G}$ is isomorphic to an induced subgraph of G; consequently, if G is bull-free and perfect, then so is $\tilde{G}$. Our next result describes how to "recover" a maximum weighted clique in G from maximum weighted cliques in $\tilde{G}$ and $G[S]$.

7.1. *Let G be a weighted graph, let S be a proper homogeneous set in G. Let $\tilde{G}$ and s be as in the definition of the reduction of G with respect to S. Let $\tilde{K}$ and K_S be maximum weighted cliques in $\tilde{G}$ and $G[S]$, respectively. If $s \notin \tilde{K}$ then set $K = \tilde{K}$, and if $s \in \tilde{K}$ then set $K = (\tilde{K} \smallsetminus \{s\}) \cup K_S$. Then K is a maximum weighted clique in G.*

Proof. First, if $s \notin \tilde{K}$ so that $K = \tilde{K}$, then it is clear that K is a clique in G and that $w_G(K) = w_{\tilde{G}}(\tilde{K})$. On the other hand, if $s \in \tilde{K}$, then $G[K]$ is obtained by substituting the complete graph $G[K_S]$ for s in the complete graph $\tilde{G}[\tilde{K}]$, and consequently K is a clique in G; furthermore, since $w_{\tilde{G}}(s) = W(G[S])$, we know that $w_G(K) = w_{\tilde{G}}(\tilde{K})$.

It remains to show that the clique K is of maximum weight in G. Let K' be a maximum weighted clique in G; we need to show that $w_G(K') \leq w_G(K)$. Let X be the set of all vertices in $V_G \smallsetminus S$ that are complete to S in G, and let Y be the set of all vertices in $V_G \smallsetminus S$ that are anti-complete to S in G. Suppose first that $K' \cap S = \emptyset$. Then K' is a clique in $\tilde{G}$ as well, and $w_{\tilde{G}}(K') = w_G(K')$; by the maximality of $\tilde{K}$, we have that:

$$w_G(K') = w_{\tilde{G}}(K') \leq w_{\tilde{G}}(\tilde{K}) = w_G(K),$$

which is what we needed to show. Suppose now that $K' \cap S \neq \emptyset$. As Y is anti-complete to S in G, and K' is a clique that intersects S in G, we know that $K' \subseteq S \cup X$. But since X is complete to S (and therefore to K_S as well) in G, and since K_S is a clique in G, we know that $(K' \smallsetminus S) \cup K_S$ is a clique in G; furthermore, since s is complete to X in $\tilde{G}$ and $K' \subseteq S \cup X$, we know that $(K' \smallsetminus S) \cup \{s\}$ is a clique in $\tilde{G}$. Now, by the maximality of $\tilde{K}$ and K_S, we have the following:

$$\begin{array}{rcl} w_G(K') & = & w_G(K' \smallsetminus S) + w_G(K' \cap S) \\ & \leq & w_G(K' \smallsetminus S) + w_G(K_S) \\ & = & w_{\tilde{G}}(K' \smallsetminus S) + w_{\tilde{G}}(s) \\ & = & w_{\tilde{G}}((K' \smallsetminus S) \cup \{s\}) \\ & \leq & w_{\tilde{G}}(\tilde{K}) \\ & = & w_G(K). \end{array}$$

This completes the argument. □

We now discuss reductions with respect to reducible homogeneous pairs. Given a weighted graph G and a reducible homogeneous pair (A, B) in G, we define below "type a reduction of G with respect to (A, B)" and "type n reduction of G with respect to (A, B)" as follows.

We first define type a reductions. Let G be a weighted graph, let (A, B) be a reducible homogeneous pair in G, and let (A, B, C, D, E, F) be the associated partition of G. Let $\tilde{G}'$ be the graph with vertex set $\{a, b, b'\} \cup C \cup D \cup E \cup F$, where a, b, and b' are pairwise distinct and do not lie in V_G, with adjacency as follows:

- $\tilde{G}'[C \cup D \cup E \cup F] = G[C \cup D \cup E \cup F]$;
- a is complete to $C \cup E$ and anti-complete to $D \cup F$;
- b and b' are complete to $D \cup E$ and anti-complete to $C \cup F$;
- a is adjacent to b and non-adjacent to b';
- b is adjacent to b'.

Next, we assign weights to the vertices of $\tilde{G}'$ as follows. The weights of the vertices in $C \cup D \cup E \cup F$ in the graph $\tilde{G}'$ are inherited from G, and for the vertices a, b, b', we set:

- $w_{\tilde{G}'}(a) = W(G[A])$;
- $w_{\tilde{G}'}(b) = W(G[A \cup B]) - W(G[A])$;
- $w_{\tilde{G}'}(b') = W(G[A]) + W([G[B]) - W(G[A \cup B])$.

We observe that all vertices in $C \cup D \cup E \cup F \cup \{a\}$ have positive integer weight in $\tilde{G}'$, b and b' have non-negative integer weights, and at most one of b and b' has zero weight. Furthermore, we have that:

- $w_{\tilde{G}'}(a) = W(G[A])$;
- $w_{\tilde{G}'}(a) + w_{\tilde{G}'}(b) = W(G[A \cup B])$;
- $w_{\tilde{G}'}(b) + w_{\tilde{G}'}(b') = W([G[B])$.

Finally, we define $\tilde{G}$ to be the graph obtained from $\tilde{G}'$ by deleting all vertices in $\tilde{G}'$ with weight zero. (Thus, either $\tilde{G} = \tilde{G}'$, or $\tilde{G} = \tilde{G}' \setminus b$, or $\tilde{G} = \tilde{G}' \setminus b'$.) We refer to the weighted graph $\tilde{G}$ as the *type* a *reduction of* G *with respect to* (A, B).

It remains to define type n reductions. Let G be a weighted graph, let (A, B) be a reducible homogeneous pair in G, and let (A, B, C, D, E, F) be the associated partition of G. Let $\tilde{G}'$ be the graph with vertex set $\{a, b, b'\} \cup C \cup D \cup E \cup F$, where a, b, and b' are pairwise distinct and do not lie in V_G, with adjacency as follows:

- $\tilde{G}'[C \cup D \cup E \cup F] = G[C \cup D \cup E \cup F]$;
- a is complete to $C \cup E$ and anti-complete to $D \cup F$;
- b and b' are complete to $D \cup E$ and anti-complete to $C \cup F$;
- a is adjacent to b and non-adjacent to b';
- b is non-adjacent to b'.

Next, we assign weights to the vertices of $\tilde{G}'$ as follows. The weights of the vertices in $C \cup D \cup E \cup F$ in the graph $\tilde{G}'$ are inherited from G, and for the vertices b, b', d, we set:

- $w_{\tilde{G}'}(a) = W(G[A])$;
- $w_{\tilde{G}'}(b) = W(G[A \cup B]) - W(G[A])$;
- $w_{\tilde{G}'}(b') = W(G[B])$.

We observe that all vertices in $C \cup D \cup E \cup F \cup \{a, b'\}$ have positive integer weight in $\tilde{G}'$, and that b has non-negative integer weight. Furthermore, we note that:

- $w_{\tilde{G}'}(a) = W(G[A])$;
- $w_{\tilde{G}'}(a) + w_{\tilde{G}'}(b) = W(G[A \cup B])$;
- $w_{\tilde{G}'}(b') = W([G[B])$.

Now, if $w_{\tilde{G}}(b) \neq 0$ then set $\tilde{G} = \tilde{G}'$, and if $w_{\tilde{G}}(b) = 0$ then set $\tilde{G} = \tilde{G}' \setminus b$; clearly, every vertex of $\tilde{G}$ has positive integer weight. We refer to the weighted graph $\tilde{G}$ as the *type* n *reduction of* G *with respect to* (A, B).

We observe that if G is a bull-free perfect graph, and (A, B) is a reducible homogeneous pair in G, then the type a reduction and the type n reduction of G with respect to (A, B) are not necessarily bull-free and perfect. We do, however, have the following result, which will suffice for the purposes of our algorithm.

7.2. *Let G be a weighted bull-free perfect graph that does not contain a proper homogneous set. Assume that applying the algorithm REDUCIBLE to G yields a 7-tuple (A, B, C, D, E, F, z). Then (A, B) is a reducible homogeneous pair in G, and the type z reduction of G with respect to (A, B) is a weighted bull-free perfect graph.*

Proof. If $z =$ a, then there exists a frame (b, b', d) for (A, B), with b adjacent to b'; and if $z =$ n, then there exists a frame (b, b', d) for (A, B), with b non-adjacent to b'. In either case, 6.3 implies that, as an unweighted graph, the reduction of type z of G with respect to (A, B) is isomorphic to an induced subgraph of G, and the result follows. □

We complete this section by describing how to "recover" a maximum weighted clique in a weighted graph G that contains a reducible homogeneous pair (A, B) from maximum weighted cliques in the weighted graphs $\tilde{G}$, $G[A]$, $G[B]$, and $G[A \cup B]$, where $\tilde{G}$ is the type a or type n reduction of the graph G with respect to (A, B).

7.3. *Let G be a weighted graph, and let (A, B) be a reducible homogeneous pair in G. Let $\tilde{G}$, $\tilde{G}'$, a, b, and b' be as in the definition of the type* a *or type* n *reduction of G with respect to (A, B). Let $\tilde{K}$, K_A, K_B, and $K_{A \cup B}$ be maximum weighted cliques in $\tilde{G}$, $G[A]$, $G[B]$, and $G[A \cup B]$, respectively. Then exactly one of the following holds:*

- *$a, b, b' \notin \tilde{K}$;*
- *$a \in K$ and $b, b' \notin \tilde{K}$;*
- *$\tilde{K}$ intersects $\{b, b'\}$, and $a \notin \tilde{K}$;*
- *$a, b \in \tilde{K}$ and $b' \notin \tilde{K}$.*

Now, define the set K as follows:

- *if $a, b, b' \notin \tilde{K}$, then set $K = \tilde{K}$;*
- *if $a \in \tilde{K}$ and $b, b' \notin \tilde{K}$, then set $K = (\tilde{K} \smallsetminus \{a\}) \cup K_A$;*
- *if $\tilde{K}$ intersects $\{b, b'\}$ and $a \notin \tilde{K}$, then set $K = (\tilde{K} \smallsetminus \{b, b'\}) \cup K_B$;*
- *if $a, b \in \tilde{K}$ and $b' \notin \tilde{K}$, then set $K = (\tilde{K} \smallsetminus \{a, b\}) \cup K_{A \cup B}$.*

Then K is a maximum weighted clique in G.

Proof. Let (A, B, C, D, E, F) be the partition of G associated with the homogeneous pair (A, B). We note that the first claim follows from the fact that a is non-adjacent to b' in $\tilde{G}'$; this also implies that the set K is well-defined.

Now, we claim that K is a clique in G, and that $w_G(K) = w_{\tilde{G}}(\tilde{K})$.

Suppose first that $a, b, b' \notin \tilde{K}$. Then $K = \tilde{K}$, and by the definition of $\tilde{G}$, we have that $G[K] = \tilde{G}[\tilde{K}]$. This implies that K is a clique in G and that $w_G(K) = w_{\tilde{G}}(\tilde{K})$.

Suppose next that $a \in \tilde{K}$ and $b, b' \notin \tilde{K}$, so that $K = (\tilde{K} \smallsetminus \{a\}) \cup K_A$. Then $G[K]$ is obtained by substituting the complete graph $G[K_A]$ for the vertex a in the complete graph $\tilde{G}[\tilde{K}]$, and so $G[K]$ is a clique; the fact that $w_G(K) = w_{\tilde{G}}(\tilde{K})$ follows from the fact that $w_{\tilde{G}}(a) = W(G[A]) = w_G(K_A)$.

Suppose now that $\tilde{K}$ intersects $\{b, b'\}$ and $a \notin \tilde{K}$, so that $K = (\tilde{K} \smallsetminus \{b, b'\}) \cup K_B$. Since $C \cup F$ is anti-complete to $\{b, b'\}$ in $\tilde{G}'$ and $\tilde{K}$ is a clique in $\tilde{G}$ (and therefore in $\tilde{G}'$ as well), we know that $\tilde{K} \subseteq \{b, b'\} \cup D \cup E$; but now since $D \cup E$ is complete to B (and therefore to K_B as well) in G, and K_B is a clique in G, it follows that K is a clique in G. It remains to show that $w_G(K) = w_{\tilde{G}}(\tilde{K})$; as $w_G(K_B) = W(G[B])$, it suffices to show that $\Sigma_{v \in \tilde{K} \cap \{b,b'\}} w_{\tilde{G}}(v) = W(G[B])$. By construction, no clique in $\tilde{G}'[b, b']$ is of weight greater than $W(G[B])$, and so $\Sigma_{v \in \tilde{K} \cap \{b,b'\}} w_{\tilde{G}}(v) \leq W(G[B])$. On the other hand, by construction, $\tilde{G}[V_{\tilde{G}} \cap \{b, b'\}]$ contains a clique $\tilde{B}$ of weight $W(G[B])$; since $\tilde{K} \subseteq \{b, b'\} \cup D \cup E$, and $D \cup E$ is complete to $\tilde{B}$ in $\tilde{G}$, we know that $(\tilde{K} \smallsetminus \{b, b'\}) \cup \tilde{B}$ is a clique in $\tilde{G}$. Since $\tilde{K}$ is of maximum weight in $\tilde{G}$, it follows that $\Sigma_{v \in \tilde{K} \cap \{b,b'\}} w_{\tilde{G}}(v) \geq w_{\tilde{G}}(\tilde{B})$. Since $\tilde{B} = W(G[B])$, this implies that $\Sigma_{v \in \tilde{K} \cap \{b,b'\}} w_{\tilde{G}}(v) = W(G[B])$. Thus, $w_G(K) = w_{\tilde{G}}(\tilde{K})$.

Finally, suppose that $a, b \in \tilde{K}$ and $b' \notin \tilde{K}$, so that $K = (\tilde{K} \smallsetminus \{a, b\}) \cup K_{A \cup B}$. Since $\tilde{K}$ is a clique in $\tilde{G}$ with $a, b \in \tilde{G}$, and since a and b are anti-complete to $C \cup F$ and $D \cup F$, respectively, in $\tilde{G}$, we get that $\tilde{K} \subseteq \{a, b\} \cup E$. But since E is complete to $A \cup B$ (and therefore to $K_{A \cup B}$ as well) in G, and since $K_{A \cup B}$ is a clique in G, it easily follows that K is a clique in G. The fact that $w_G(K) = w_{\tilde{G}}(\tilde{K})$ follows from the fact that $w_{\tilde{G}}(a) + w_{\tilde{G}}(b) = W(G[A \cup B]) = w_G(K_{A \cup B})$.

It remains to show that the clique K is of maximum weight in G. Fix some maximum weighted clique K' in G; we need to show that $w_G(K') \leq w_G(K)$.

Suppose first that $K' \cap (A \cap B) = \emptyset$. Then K' is a clique in $\tilde{G}$, and so by the maximality of $\tilde{K}$, we have that $w_{\tilde{G}}(K') \leq w_{\tilde{G}}(\tilde{K}) = w_G(K)$, which is what we needed to show.

Suppose next that $K' \cap A \neq \emptyset$ and $K' \cap B = \emptyset$. Since K' is a clique and $D \cup F$ is anti-complete to A in G, we have that $K' \subseteq A \cup C \cup E$. Since $C \cup E$ is complete to A in G, and since $K_A \subseteq A$, we know that $(K' \smallsetminus A) \cup K_A$ is a clique in G and that $(K' \smallsetminus A) \cup \{a\}$ is a clique in $\tilde{G}$. Now by the maximality of K_A and $\tilde{K}$, we get the following:

$$\begin{aligned}
w_G(K') &= w_G(K' \smallsetminus A) + w_G(K' \cap A) \\
&\leq w_G(K' \smallsetminus A) + w_G(K_A) \\
&= w_{\tilde{G}}(K' \smallsetminus A) + w_{\tilde{G}}(a) \\
&= w_{\tilde{G}}((K' \smallsetminus A) \cup \{a\}) \\
&\leq w_{\tilde{G}}(\tilde{K}) \\
&= w_G(K).
\end{aligned}$$

Next, suppose that $K' \cap B \neq \emptyset$ and $K' \cap A = \emptyset$. Since K' is a clique and $C \cup F$ is anti-complete to B, we have that $K' \subseteq B \cup D \cup E$. Since $D \cup E$ is complete to B in G, and since $K_B \subseteq B$, we know that $(K' \smallsetminus B) \cup K_B$ is a clique in G. By the construction of $\tilde{G}$, $\tilde{G}[V_{\tilde{G}} \cap \{b, b'\}]$ contains a clique $\tilde{B}$ of weight $W(G[B]) = w_G(K_B)$; clearly, $\tilde{B}$ is complete to $D \cup E$ in $\tilde{G}$, and so it easily follows that $(K' \smallsetminus B) \cup \tilde{B}$ is a clique in $\tilde{G}$. Now, by the maximality of K_B and $\tilde{K}$, we have the following:

$$\begin{aligned}
w_G(K') &= w_G(K' \smallsetminus B) + w_G(K' \cap B) \\
&\leq w_G(K' \smallsetminus B) + w_G(K_B) \\
&= w_{\tilde{G}}(K' \smallsetminus B) + w_{\tilde{G}}(\tilde{B}) \\
&= w_{\tilde{G}}((K' \smallsetminus B) \cup \tilde{B}) \\
&\leq w_{\tilde{G}}(\tilde{K}) \\
&= w_G(K).
\end{aligned}$$

Suppose, finally, that K' intersects both A and B. As $C \cup F$ is anti-complete to B, and $D \cup F$ is anti-complete to A, we have that $K' \subseteq A \cup B \cup E$. As E is complete to $A \cup B$ (and therefore to $K_{A \cup B}$ as well), we know that $(K' \smallsetminus (A \cup B)) \cup K_{A \cup B}$ is a clique in G. Now, set $H = V_{\tilde{G}} \cap \{a, b\}$; clearly, H is a clique complete to E in $\tilde{G}$, and so $(K' \smallsetminus (A \cup B)) \cup H$ is a clique in $\tilde{G}$. Furthermore, note that $w_{\tilde{G}}(H) = W(G[A \cup B]) = w_G(K_{A \cup B})$. Now, by the maximality of $K_{A \cup B}$ and $\tilde{K}$, we have the following:

$$\begin{aligned}
w_G(K') &= w_G(K' \smallsetminus (A \cup B)) + w_G(K' \cap (A \cup B)) \\
&\leq w_G(K' \smallsetminus (A \cup B)) + w_G(K_{A \cup B}) \\
&= w_{\tilde{G}}(K' \smallsetminus (A \cup B)) + w_{\tilde{G}}(H) \\
&= w_{\tilde{G}}((K' \smallsetminus (A \cup B)) \cup H) \\
&\leq w_{\tilde{G}}(\tilde{K}) \\
&= w_G(K).
\end{aligned}$$

This completes the argument. □

8 The Algorithm

In this section, we describe the algorithm MWCLIQUE whose input is a weighted bull-free perfect graph G, and whose output is a maximum weighted clique in G. We begin by discussing some previously known algorithms that we use in our algorithm MWCLIQUE.

First, given a graph G on n vertices, one can use the algorithm from [16] or the algorithm from [23] to check whether G is transitively orientable, and if so, to find a transitive orientation for G; this takes at most $O(n^3)$ time. Next, given a weighted transitive directed graph G on n vertices, one can use the algorithm from [18] to find a maximum weighted clique in G in at most $O(n^3)$ time, and one can use the algorithm from [2] to find a maximum weighted stable set in G (which is a maximum weighted clique in $\overline{G}$) in at most $O(n^4)$ time. (In fact, in [2], the problem of finding a maximum weighted stable set in a weighted transitive directed graph is reduced to finding a maximum weighted stable set in a weighted bipartite graph. The latter can be done using network flows, as explained, for example, in section 2 of [13].) All of this implies that, given a weighted graph G on n vertices, one can determine whether at least one of G and $\overline{G}$ is transitively orientable in at most $O(n^3)$ time, and if so, one can find a maximum weighted clique in G in at most $O(n^4)$ time.

Second, given a graph G on n vertices, one can use an algorithm from any one of [11], [12], or [24] to check whether G has a proper homogeneous set, and if so, to find a proper homogeneous set in G. This takes at most $O(n^2)$ time.

We now turn to describing the algorithm MWCLIQUE. As stated at the beginning of this section, the input is a weighted bull-free perfect graph G, and the output is a maximum weighted clique in G. Along with the algorithm, we construct a rooted decomposition tree T_G associated with G. The vertices of T_G are the graphs constructed by the algorithm, and the root of T_G is the graph G. In this paper, a *leaf* of a rooted tree is a vertex of the tree that has no descendants. (In particular, every non-root vertex of degree one in a rooted tree is a leaf, and the root is a leaf if and only if the tree consists of the root only.)

Suppose that G is a weighed bull-free perfect graph, and set $n = |V_G|$. By 2.1, at least one of the following holds:

- G or $\overline{G}$ is transitively orientable;
- G contains a proper homogeneous set;
- G contains a reducible homogeneous pair.

The first step is to check whether at least one of G and $\overline{G}$ is transitively orientable, and if so, to find a maximum weighted clique in G; as explained above, this takes at most $O(n^4)$ time. In this case, G is a leaf of the rooted tree T_G.

From now on, we assume that neither G nor $\overline{G}$ is transitively orientable. We then check whether G contains a proper homogeneous set, and if so, we find a proper homogeneous set in G; as explained above, this takes at most $O(n^2)$ time. If the algorithm returns a proper homogeneous set S, then we call the algorithm MWCLIQUE on the graphs $G[S]$ and $\tilde{G}$, where $\tilde{G}$ is the reduction of G with respect to S, as defined in section 7. Once we have obtained maximum weighted cliques for $G[S]$ and $\tilde{G}$, we can find a maximum weighted clique in G as outlined in 7.1. In this case, G has two children in the tree T_G, namely $G[S]$ and $\tilde{G}$.

From now on, we assume that G does not contain a proper homogeneous set. Then G contains a reducible homogeneous pair. We now call the algorithm REDUCIBLE from section 6 on the graph G, and obtain a 7-tuple (A, B, C, D, E, F, z), where (A, B) is a reducible homogeneous pair in G, (A, B, C, D, E, F) is the partition of G associated with (A, B), and $z \in \{\text{a}, \text{n}\}$; this takes at most $O(n^5)$ time. By 2.2, $G[A]$ and $G[B]$ are both transitively orientable, and so (as explained above) we can find a maximum weighted clique in each of them in at most $O(n^3)$ time. We then call the algorithm MWCLIQUE on the graphs $G[A \cup B]$ and $\tilde{G}$, where $\tilde{G}$ is the type z reduction of G with respect to (A, B), as defined in section 7; we note that the graph $\tilde{G}$ is bull-free and perfect by 7.2, and since $|A \cup B| \geq 4$, we know that $\tilde{G}$ has fewer vertices than G. Once we have obtained maximum weighted cliques for each of $G[A]$, $G[B]$, $G[A \cup B]$, and $\tilde{G}$, we can find a maximum weighted clique in G as outlined in 7.3. In this case, G has two children in the tree T_G, namely $G[A \cup B]$ and $\tilde{G}$.

9 Complexity Analysis

Our goal in this section is to prove the following result.

9.1. *The running time of the algorithm MWCLIQUE is at most $O(n^6)$, where n is the number of vertices of the input graph.*

We observe that each step of the algorithm MWCLIQUE can be performed in at most $O(n^5)$ time, and so in order to prove 9.1, it suffices to show that for each weighted bull-free perfect graph G, the number of vertices in the decomposition tree T_G is bounded by a linear function of the number of vertices of G. We begin with a technical lemma (9.2), and then we use this lemma to prove 9.3, which states that the number of vertices in the decomposition tree T_G of a weighted bull-free perfect graph G is at most $3|V_G|$. The main result of this section (9.1) then follows immediately.

9.2. *Let T be a rooted tree with root r. Let $f : V_T \to \mathbb{N}$ be a function such that for all vertices $v \in V_T$ that are not leaves of T, if $v_1, ..., v_k \in V_T$ are the children of v, then $\Sigma_{i=1}^{k} f(v_k) < f(v)$. Then $|V_T| \leq f(r)$.*

Proof. We proceed by induction on $|V_T|$. If $|V_T| = 1$, then the result is immediate as $f(r)$ is a positive integer. So assume that T has at least two vertices, and that the claim holds for rooted trees with fewer vertices. Let $r_1, ..., r_k$ be the children of the root r in the tree T. For each $i \in \{1, ..., k\}$, let T_i be the component of $T \setminus r$ that contains r_i; we turn T_i into a rooted tree by letting r_i be the root of T_i. By the induction hypothesis, $|V_{T_i}| \leq f(r_i)$ for each $i \in \{1, ..., k\}$. But then note the following:

$$\begin{aligned} |V_T| &= 1 + \Sigma_{i=1}^{k} |V_{T_i}| \\ &\leq 1 + \Sigma_{i=1}^{k} f(r_i) \\ &< 1 + f(r). \end{aligned}$$

Since $|V_T|$ and $f(r)$ are both integers, it follows that $|V_T| \leq f(r)$, as we had claimed. □

9.3. *Let G be a weighted bull-free perfect graph, and let $n = |V_G|$. Then the number of vertices in the decomposition tree T_G is at most $3n$.*

Proof. If G or $\overline{G}$ is transitively orientable, then the tree T_G has only one vertex (namely, the root G), and the result is immediate. So assume that G and $\overline{G}$ are not transitively orientable. It is easy to check that every graph on at most four vertices is transitively orientable; consequently, $n \geq 5$. Furthermore, G has exactly two children in the tree T_G, and so in particular, the root G is not a leaf of T_G. Now, let T'_G be the graph obtained from T_G by deleting all the leaves of T_G. As every vertex of T_G has at most two children, it follows that $|V_{T_G}| \leq 3|V_{T'_G}|$. Thus, in order to show that $|V_{T_G}| \leq 3n$, we just have to show that $|V_{T'_G}| \leq n$. Note that no vertex H of T'_G is transitively orientable, for otherwise, H would be a leaf of T_G, contrary to the fact that T'_G contains no leaves of T_G; since every graph on at most four vertices is transitively orientable, it follows that every vertex of T'_G has at least five vertices. Now, to each vertex H of T'_G, we associate the number $f(H) = |V_H| - 4$; since every vertex in the tree T'_G has at least five vertices, it follows that $f(H)$ is a positive integer for every vertex H of T'_G.

Suppose that H is not a leaf of T'_G, and let $H_1, ..., H_k$ be the children of H in T'_G; we claim that $\Sigma_{i=1}^{k} f(H_i) < f(H)$. By construction, every child of H has fewer vertices than H, and so if $k = 1$, the result is immediate. So assume that $k \geq 2$; as every vertex in T_G that is not a leaf has exactly two children, it follows that $k = 2$, and we need to show that $f(H_1) + f(H_2) < f(H)$. If H contains a proper homogeneous set, then we may assume that $|V_{H_1}| = p$,

$|V_{H_2}| = q+1$, and $|V_H| = p+q$. If H does not contain a proper homogeneous set, then H contains a reducible homogeneous pair, and we may assume that $|V_{H_1}| = p$, $|V_{H_2}| = q+2$ or $|V_{H_2}| = q+3$, and $|V_H| = p+q$. In any case, we may assume that $|V_{H_1}| = p$, $|V_{H_2}| = q+r$, and $|V_H| = p+q$, for some positive integers p and q, and some $r \in \{1,2,3\}$. But then we have the following:

$$\begin{aligned} f(H_1) + f(H_2) &= (p-4) + (q+r-4) \\ &= p+q+r-8 \\ &< p+q-4 \\ &= f(H) \end{aligned}$$

But now 9.2 implies that T'_G has at most $f(G) = n-4$ vertices, which completes the argument. □

We now restate and prove the main result of the section.

9.1. *The running time of the algorithm MWCLIQUE is at most $O(n^6)$, where n is the number of vertices of the input graph.*

Proof. Each step of the algorithm takes at most $O(n^5)$ time, and by 9.3, we make at most $O(n)$ calls to the algorithm. The result is then immediate. □

10 A Comparison with the Algorithm from [14]

It is natural to ask why the algorithm MWCLIQUE is faster than the algorithm from [14] due to de Figueiredo and Maffray. In this final section, we discuss some reasons for this. We remind the reader that both the algorithm from the present paper and the algorithm from [14] find a maximum weighted clique in a weighted bull-free perfect graph, and then, relying on the argument outlined in the Introduction, use this algorithm in order to solve the following three optimization problems: the maximum weighted stable set problem, the minimum weighted coloring problem, and the minimum weighted clique covering problem. In what follows, we compare the two algorithms (the one from the present paper and the one from [14]) that find a maximum weighted clique in a weighted bull-free perfect graph.

In this section, n denotes the number of vertices and m the number of edges of the input graph. First, we remind the reader that the running time of the algorithm from [14] is $O(n^5m^3)$; the slowest step of the algorithm takes $O(n^4m)$ time, and there are at most $O(nm^2)$ recursive calls to the algorithm. In comparison, the running time of the algorithm MWCLIQUE is $O(n^6)$; the slowest step of this algorithm takes $O(n^5)$ time, and there are at most $O(n)$ recursive calls to the algorithm. Below, we discuss the reasons why the algorithm from [14] is slower than the one from the present paper, and we also explain how the complexity of the algorithm from [14] could be improved by using some ideas from the present paper.

We first need a couple of definitions. A graph G is said to be *weakly triangulated* provided neither G nor $\overline{G}$ contains an induced cycle of length at least five. Next, let (A, B) be a homogeneous pair in a graph G, and let (A, B, C, D, E, F) be the associated partition of G; we say that (A, B) is *interesting* provided that all of the following hold:

- $G[A]$ and $G[B]$ are both connected;
- G contains an induced cycle $a - a' - b' - b - a$ such that $a, a' \in A$ and $b, b' \in B$;
- C and D are both non-empty;
- C is not anti-complete to D.

We remark that all interesting homogeneous pairs are reducible, but the reverse does not hold. We now state a decomposition theorem that the algorithm from [14] is based on. (We remark that this theorem is not explicitly stated this way in [14]; see section 6 of [14]).

10.1 (de Figueiredo and Maffray [14]). *Let G be a bull-free perfect graph. Then at least one of the following holds:*

- *G contains a proper homogeneous set;*
- *G or $\overline{G}$ is transitively orientable;*
- *G is weakly triangulated;*
- *G contains an interesting homogeneous pair.*

The difference between 10.1 and the decomposition theorem 2.1 derived in the present paper is, first, that 10.1 involves weakly triangulated graphs while 2.1 does not, and second, that the homogeneous pairs from the two theorems are different. The slowest step of the algorithm from [14] is the one that finds a maximum weighted clique in a weakly triangulated graph; this step takes $O(n^4 m)$ time. This step does not appear in the algorithm MWCLIQUE; the slowest step of the algorithm MWCLIQUE takes $O(n^5)$ time.

We next discuss the number of recursive calls that the algorithm from [14] makes. If the algorithm finds an interesting homogeneous pair (A, B) in the input graph G, it makes recursive calls to two graphs: $Q = G[A \cup B]$, and a graph H obtained by adding four new vertices to the graph $G \setminus (A \cup B)$ (we omit the details and refer the reader to [14]). Note that $A \cup B$ may contain as few as four vertices, and consequently, the graph H may possibly have the same number of vertices as the original graph G. Now the authors of

[14] argue that Q and H together contain fewer induced two-edge matchings than G does, and ultimately, the algorithm makes at most $O(nm^2)$ recursive calls to the algorithm. In comparison, the algorithm MWCLIQUE from the present paper makes only $O(n)$ recursive calls.

Finally, we remark that since every interesting homogeneous pair is reducible, the algorithm from [14] could be changed so that instead of the graph H (which introduces four new vertices), the reductions introduced in section 7 of the present paper are used. This would reduce the number of recursive calls to the algorithm to $O(n)$, and consequently, the running time of the algorithm from [14] for finding a maximum weighted clique in a weighted bull-free perfect graph would drop from $O(n^5m^3)$ to $O(n^5m)$.

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