Coloring $(4K_1, C_4, C_6)$-free graphs

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Abstract

$(4K_1, C_4, C_6)$-free graphs are precisely the even-hole-free graphs of stability number at most three. We show that $(4K_1, C_4, C_6)$-free graphs can be recognized in $O(n^3)$ time, and furthermore, that all the following can be computed in $O(n^3)$ time for such graphs: an optimal coloring, a minimum clique cover, and the list of all maximal cliques. We also show that every $(4K_1, C_4, C_6)$-free graph on n vertices has at most n maximal cliques.

Keywords: $(4K_1, C_4, C_6)$-free graphs; even-hole-free graphs; graph coloring; maximal cliques; minimum clique cover

1 Introduction

All graphs in this paper are finite, simple, and nonnull. As usual, the vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. In all our algorithms, n is the number of vertices and m the number of edges of the input graph.

For a positive integer k, K_k is the complete graph on k vertices, and C_k (for $k \geq 3$) is the cycle on k vertices. $4K_1$ is the edgeless graph on four vertices. For a graph H, a graph G is said to be *H-free* if no induced subgraph of G is isomorphic to H. For a family of graphs $\mathcal{H}$, a graph G is said to be *$\mathcal{H}$-free* if G is H-free for all $H \in \mathcal{H}$. A *hole* in a graph G is an induced cycle in G of length at least four. A hole is *even* or *odd* depending on the parity of its length.

A *clique* of a graph G is a (possibly empty) set of pairwise adjacent vertices of G; a clique C of G is *maximal* if no clique of G contains C as a proper subset. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique of G. A *stable set* of G is a (possibly empty) set of pairwise nonadjacent vertices of G; the *stability number* of G, denoted by $\alpha(G)$, is the maximum size of a stable set of G. Note that a graph G is $4K_1$-free if and only if $\alpha(G) \leq 3$.

A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. For an integer k, a graph G is said to be *k-colorable* if there exists a proper coloring of G that uses at most k colors. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable. An *optimal coloring* of G is a proper coloring of G that uses only $\chi(G)$ colors.

A *clique cover* of a graph G is a collection of cliques of G such that every vertex of G belongs to at least one of those cliques. The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the minimum number of cliques in any clique cover of G. A *minimum clique cover* of G is a clique cover of G that consists of only $\overline{\chi}(G)$ cliques.

Even-hole-free graphs are a well-studied class of graphs, partly due to their structural similarity with perfect graphs (for a structural description of even-hole-free graphs, see [8, 11]).[1]

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[1] A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Perfect graphs can be recognized in polynomial time [6],[2] and furthermore, the following four optimization problems can all be solved in polynomial time for perfect graphs: MAXIMUM CLIQUE, MAXIMUM STABLE SET, GRAPH COLORING (also called VERTEX COLORING), and CLIQUE COVER [18]. Even-hole-free graphs can also be recognized in polynomial time [5, 9], but of the four optimization problems mentioned above, only the MAXIMUM CLIQUE problem is known to be solvable in polynomial time for even-hole-free graphs [10]. More precisely, it was shown in [10] that every even-hole-free graph G has at most $|V(G)| + 2|E(G)|$ maximal cliques, and they can all be computed in $O(n^2m)$ time; this immediately implies that the MAXIMUM CLIQUE problem can be solved in $O(n^2m)$ time for even-hole-free graphs.[3] The time complexity of the remaining three optimization problems (namely, MAXIMUM STABLE SET, GRAPH COLORING, and CLIQUE COVER) is unknown for the class of even-hole-free graphs.

Another class for which the time complexity of GRAPH COLORING remains open is that of $(4K_1, C_4)$-free graphs. Foley *et al.* [14] asked whether GRAPH COLORING can be solved in polynomial time for graphs in the intersection of these two classes, that is, for even-hole-free graphs of stability number at most three, or equivalently, for $(4K_1, C_4, C_6)$-free graphs.[4] In [23], the author of the present paper showed that this question has a positive answer. Let us discuss this in more detail. First, Foley *et al.* [14] proved a structure theorem for $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7, and they showed that such graphs have bounded clique-width, and that GRAPH COLORING can therefore (by [24]) be solved in polynomial time for such graphs. We note that Koutecký [20] recently constructed an $O(n + m)$ time coloring algorithm for $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7; this algorithm relies on the structural results from [14] and on integer programming. Further, the author of the present paper proved a structure theorem for $(4K_1, C_4, C_6, C_7)$-free graphs that contain no simplicial vertices,[5] and used this theorem to show that such graphs have bounded clique-width [23]. Thus, $(4K_1, C_4, C_6)$-free graphs that contain no simplicial vertices have bounded clique-width. Simplicial vertices pose no obstacle to coloring in polynomial time, and GRAPH COLORING can be solved in polynomial time for graphs of bounded clique-width [24]; thus, GRAPH COLORING can be solved in polynomial time for $(4K_1, C_4, C_6)$-free graphs (see Corollary 1.4 of [23]). However, all known polynomial-time coloring algorithms for graphs of clique-width at most k have running time $O(n^{f(k)})$ for some fast growing function f (see [15] for an overview). Thus, it is of interest to construct a faster coloring algorithm for $(4K_1, C_4, C_6)$-free graphs.

Clearly, $(4K_1, C_4, C_6)$-free graphs can be recognized in $O(n^6)$ time. In the present paper, we use the structural results from [14, 23] to give an $O(n^3)$ time recognition algorithm for $(4K_1, C_4, C_6)$-free graphs. We remark that our recognition algorithm may possibly (correctly) determine that the input graph is not $(4K_1, C_4, C_6)$-free, without actually detecting any of the three forbidden induced subgraphs in it; indeed, it may happen that the algorithm simply determines that the input graph does not have the structure described in [14, 23], and is therefore not $(4K_1, C_4, C_6)$-free. Further, we show that any $(4K_1, C_4, C_6)$-free graph G has at most $|V(G)|$ maximal cliques, and we construct an $O(n^3)$ time algorithm that computes them all;

[2]The algorithm from [6] is in fact a recognition algorithm for "Berge" graphs. A graph is *Berge* if neither it nor its complement contains an odd hole. By the Strong Perfect Graph Theorem [7], a graph is perfect if and only if it is Berge.

[3]More generally, any C_4-free graph has only $O(n^2)$ maximal cliques [1, 13], and if a graph has K maximal cliques, they can all be found in $O(Kn^3)$ time by combining results from [22, 27]. So, all maximal cliques of a C_4-free graph can be found in $O(n^5)$ time. In particular, the MAXIMUM CLIQUE problem can be solved in $O(n^5)$ time for C_4-free graphs.

[4]Obviously, even-hole-free graphs of stability number at most three are $(4K_1, C_4, C_6)$-free. Conversely, every hole of length at least eight contains an induced $4K_1$, and so all holes in a $(4K_1, C_4, C_6)$-free graph are of length five or seven; in particular, $(4K_1, C_4, C_6)$-free graphs are even-hole-free, and obviously, their stability number is at most three.

[5]A *simplicial vertex* is a vertex whose neighborhood is a (possibly empty) clique.

this immediately implies that the Maximum Clique problem can be solved in $O(n^3)$ time for $(4K_1, C_4, C_6)$-free graphs. Finally, we give $O(n^3)$ time algorithms that compute an optimal coloring and a minimum clique cover of a $(4K_1, C_4, C_6)$-free graph. All our results rely on the structural description of $(4K_1, C_4, C_6)$-free graphs from [14, 23]; furthermore, all our algorithms are purely combinatorial. We remark that our coloring algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs also relies on the recently obtained formula for the chromatic number of a "ring" [21], and our coloring algorithm for $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7 uses the coloring algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs as a subroutine. The reader will have noticed that we could, instead, have used the $O(n+m)$ time algorithm from [20] to color $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7. However, the algorithm from [20] relies on some fairly sophisticated integer programming results, and it is therefore not combinatorial. Since our coloring algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs runs in $O(n^3)$ time, the overall running time of our coloring algorithm for $(4K_1, C_4, C_6)$-free graphs would still be $O(n^3)$, even if we relied on the algorithm from [20]. Finally, we remark that a maximum stable set of a $(4K_1, C_4, C_6)$-free graph can trivially be found in $O(n^3)$ time, since all stable sets of a $(4K_1, C_4, C_6)$-free graph are of size at most three.

We complete the introduction with an outline of the paper. In section 2, we introduce some (mostly standard) terminology and notation that we will use throughout the paper. In section 3, we state some results from the literature that we will need for our algorithms. In section 4, we describe our $O(n^3)$ time recognition algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs (see Theorem 4.4). In section 5, we construct $O(n^3)$ time algorithms that compute all maximal cliques and a minimum vertex cover of a $(4K_1, C_4, C_6, C_7)$-free graph (see Theorems 5.1 and 5.4, respectively). In section 6, we give an $O(n^3)$ time coloring algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs (see Theorem 6.8). In section 7, we deal with $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7 (see Theorems 7.1, 7.4, and 7.8). Finally, in section 8, we prove the main results of this paper (see Theorems 8.1 and 8.2).

2 Terminology and notation

$\mathbb{N}$ is the set of all nonnegative integers. For a set S of sets, we write $\bigcup S := \bigcup_{X \in S} X$. A *singleton* is a set that has exactly one element.

For a vertex x in a graph G, the *open neighborhood* (or simply *neighborhood*) of x in G, denoted by $N_G(x)$, is the set of all neighbors of x in G, and the *closed neighborhood* of x in G, denoted by $N_G[x]$, is defined as $N_G[x] = \{x\} \cup N_G(x)$. The *degree* of x in G, denoted by $d_G(x)$, is the number of neighbors that x has in G, *i.e.* $d_G(x) = |N_G(x)|$. A *nonneighbor* of x is any vertex in $V(G) \setminus \{x\}$ that is nonadjacent to x,[6] and the *nonneighborhood* of x in G is the set $V(G) \setminus N_G[x]$.

Given a graph G and distinct vertices $x, y \in V(G)$, we say that x *dominates* y in G, or that y is *dominated* by x in G, provided that $N_G[y] \subseteq N_G[x]$.[7] A vertex $v \in V(G)$ is *universal* in G if v is adjacent to all other vertices of G, *i.e.* if $N_G[v] = V(G)$.

For a graph G and a nonempty set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S; for vertices $x_1, \dots, x_t \in V(G)$, we sometimes write $G[x_1, \dots, x_t]$ instead of $G[\{x_1, \dots, x_t\}]$. For a set $S \subsetneqq V(G)$, $G \setminus S$ is the subgraph of G obtained by deleting S, *i.e.* $G \setminus S = G[V(G) \setminus S]$. If G has at least two vertices and $x \in V(G)$, we sometimes write $G \setminus x$ instead of $G \setminus \{x\}$.[8]

For an integer $k \geq 4$, a *k-hole* in a graph G is an induced C_k in G. When we write "$x_0, \dots, x_{k-1}, x_0$ is a k-hole in G," we mean that $x_0, \dots, x_{k-1}$ are pairwise distinct vertices of G, and the edges of $G[x_0, \dots, x_{k-1}]$ are precisely $x_0x_1, x_1x_2, x_2x_3, \dots, x_{k-2}x_{k-1}, x_{k-1}x_0$.

[6] So, x is neither a neighbor nor a nonneighbor of itself.

[7] In particular, x and y must be adjacent.

[8] Since our graphs are nonnull, if $V(G) = \{x\}$, then $G \setminus x$ is not defined.

Given a graph G, a vertex $x \in V(G)$, and a set $Y \subseteq V(G) \setminus \{x\}$, we say that x is *complete* (resp. *anticomplete*) to Y in G provided that x is adjacent (resp. nonadjacent) to all vertices of Y in G.

Given a graph G and disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (resp. *anticomplete*) to Y in G provided that every vertex in X is complete to Y in G.

As usual, the *complement* of a graph G, denoted by $\overline{G}$, is the graph whose vertex set is $V(G)$ and in which two distinct vertices are adjacent if and only if they are nonadjacent in G. A graph is *anticonnected* if its complement is connected. Obviously, every anticonnected graph on at least two vertices contains a pair of nonadjacent vertices.

An *anticomponent* of a graph G is an induced subgraph H of G such that $\overline{H}$ is a connected component of $\overline{G}$. An anticomponent is *trivial* if it has only one vertex, and it is *nontrivial* if it has at least two vertices. Clearly, the vertex sets of the anticomponents of a graph G are complete to each other in G. Thus, when we say that "Q is the only nontrivial anticomponent of G," we have that Q is an anticonnected induced subgraph of G, that $|V(Q)| \geq 2$, and that $V(G) \setminus V(Q)$ is a (possibly empty) clique, complete to $V(Q)$ in G; in particular, G can be obtained from Q by repeatedly adding universal vertices (possibly, $G = Q$).

2.1 A convention for figures

In all our figures, crosshatched disks represent cliques. A disk or rectangle that is not crosshatched represents a set of vertices, which need not be a clique. A straight line between two regions (two disks, two rectangles, or a disk and a rectangle) indicates that the corresponding sets of vertices are complete to each other. A wavy line between two regions indicates that there may be edges between the corresponding sets of vertices.[9] The absence of a line (straight or wavy) between two regions indicates that the corresponding sets of vertices are anticomplete to each other.

3 Some results from the literature

3.1 Simplicial vertices and chordal graphs

A *chordal graph* is a graph that does not contain any holes. A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Theorem 3.1. *[2, 12] Chordal graphs are perfect. In particular, every chordal graph G satisfies $\chi(G) = \omega(G)$.*

A vertex x in a graph G is *simplicial* if $N_G(x)$ is a (possibly empty) clique of G. A *simplicial elimination ordering* of a graph G is an ordering $v_1, \dots, v_n$ of its vertices such that for all $i \in \{1, \dots, n\}$, v_i is simplicial in the graph $G[v_i, v_{i+1}, \dots, v_n] = G \setminus \{v_1, \dots, v_{i-1}\}$.

Theorem 3.2. *[16] A graph is chordal if and only if it admits a simplicial elimination ordering.*

By examining the neighborhood of each vertex, one can find a simplicial vertex in the input graph, or determine that the graph contains no simplicial vertices, in $O(n^3)$ time. By repeating the process $O(n)$ times, one can find a maximal sequence $v_1, \dots, v_s$ ($s \geq 0$) of pairwise distinct vertices of G such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$; this takes a total of $O(n^4)$ time. It is, however, possible to find the sequence $v_1, \dots, v_s$ slightly faster, in only $O(n^3)$ time (see Lemma 3.3 below). As stated, the algorithm from Lemma 3.3 is taken from [21] (see Lemma 2.5 of [21]). However, as explained in [21], this algorithm is only a minor modification of an algorithm described in the introduction of [19].

[9] However, wavy lines do not indicate that adjacency is arbitrary; it must have the properties specified in the text.

Lemma 3.3. *[19, 21] There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *A maximal sequence $v_1, \dots, v_s$ ($s \geq 0$) of pairwise distinct vertices of G such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$;*[10]
- ***Running time:*** *$O(n^3)$.*

We will use Lemma 3.3 repeatedly in our algorithms. We remark that Theorem 3.2 and Lemma 3.3 together yield an $O(n^3)$ time recognition algorithm for chordal graphs.[11] In fact, chordal graphs can be recognized in $O(n+m)$ time using lexicographic breadth-first-search [25]; however, we will not use this faster (but more complicated) algorithm, since it would not improve the overall running time of any of our algorithms.

Gavril [17] showed that, given an input chordal graph G, together with a simplicial elimination ordering $v_1, \dots, v_n$, the stability number of G can be computed in $O(n+m)$ time, which is $O(n^2)$ time. We state this below for future reference.

Lemma 3.4. *[17] There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G and a simplicial elimination ordering of G;*[12]
- ***Output:*** *$\alpha(G)$;*
- ***Running time:*** *$O(n^2)$.*

3.2 True twins

Given a graph G, two distinct vertices $u, v \in V(G)$ are said to be *true twins* in G if $N_G[u] = N_G[v]$. Clearly, the relation of being a true twin is an equivalence relation; a *true twin class* of G is an equivalence class with respect to the true twin relation.[13] Thus, $V(G)$ can be partitioned into true twin classes of G in a unique way, and clearly, every true twin class of G is a clique of G. Note that two distinct true twin classes of G are either complete or anticomplete to each other in G. An exercise from [26] states that, given an input graph G, all true twin classes of G can be found in $O(n+m)$ time, which is $O(n^2)$ time; a detailed proof of this result can be found in [4]. Given a graph G and a partition $\mathcal{P}$ of $V(G)$ into true twin classes of G, we define the graph $G_{\mathcal{P}}$ (called the *quotient graph* of G with respect to $\mathcal{P}$) to be the graph whose vertex set is $\mathcal{P}$, and in which distinct $A, B \in \mathcal{P}$ are adjacent if and only if A and B are complete to each other in G. Clearly, given G and $\mathcal{P}$, the graph $G_{\mathcal{P}}$ can be found in $O(n^2)$ time. We summarize these results below for future reference.

Lemma 3.5. *[4, 26] There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *The partition $\mathcal{P}$ of $V(G)$ into true twin classes of G, and the quotient graph $G_{\mathcal{P}}$;*
- ***Running time:*** *$O(n^2)$.*

[10] If $s = 0$, then the sequence $v_1, \dots, v_s$ is empty and G has no simplicial vertices.

[11] Indeed, suppose that for an input graph G, the $O(n^3)$ time algorithm from Lemma 3.3 produces a sequence $v_1, \dots, v_s$. If $V(G) = \{v_1, \dots, v_s\}$, then $v_1, \dots, v_s$ is a simplicial elimination ordering of G, and so Theorem 3.2 guarantees that G is chordal. On the other hand, if $\{v_1, \dots, v_s\} \subsetneqq V(G)$, then the maximality of $v_1, \dots, v_s$ guarantees that $G \setminus \{v_1, \dots, v_s\}$ has no simplicial vertices, and consequently, does not admit a simplicial elimination ordering; in this case, Theorem 3.2 guarantees that $G \setminus \{v_1, \dots, v_s\}$ is not chordal, and it follows that G is not chordal, either.

[12] By Theorem 3.2, this means that G is chordal.

[13] Technically, the equivalence relation in question is the "is a true twin of or is equal to" relation. (A vertex is not a true twin of itself.)

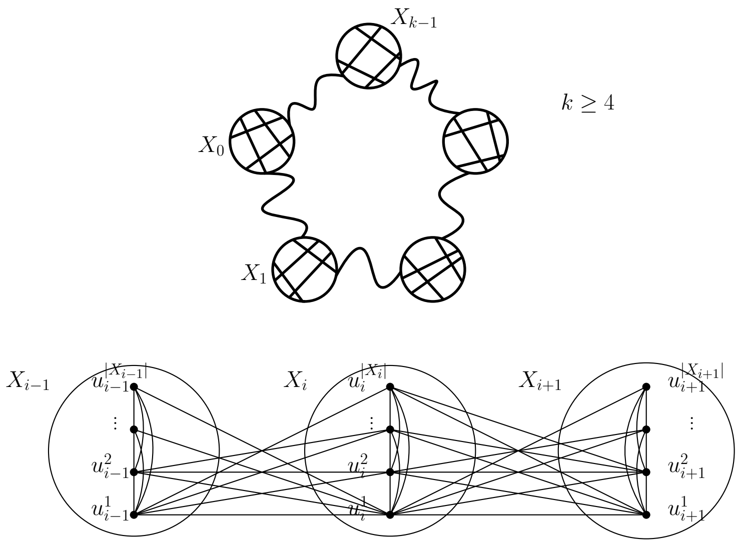


Figure 3.1: Top: A k-ring R ($k \geq 4$) with ring partition $(X_0, \dots, X_{k-1})$. Bottom: The subgraph $R[X_{i-1} \cup X_i \cup X_{i+1}]$ ($i \in \mathbb{Z}_k$); note that u_i^1 is complete to $X_{i-1} \cup X_{i+1}$.

3.3 Rings and hyperholes

A *ring* is a graph R whose vertex set can be partitioned into $k \geq 4$ nonempty sets, $X_0, \dots, X_{k-1}$ (with indices understood to be in $\mathbb{Z}_k$), such that for all $i \in \mathbb{Z}_k$, the set X_i can be ordered as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$.[14] Under these circumstances, we also say that R is a *k-ring* or a ring of *length* k; furthermore, we say that $(X_0, \dots, X_{k-1})$ is a *ring partition* of the ring R. (See Figure 3.1.)

Rings were originally introduced in [3]. They can be recognized in polynomial (quadratic) time; more precisely, the following is Lemma 8.14 from [3].

Lemma 3.6. *[3] There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *Either the true statement that G is a ring, together with the length and ring partition of the ring, or the true statement that G is not a ring;*
- ***Running time:*** *$O(n^2)$.*

The following is Lemma 2.4(a)-(c) from [3].

Lemma 3.7. *[3] Let R be a k-ring with ring partition $(X_1, \dots, X_k)$. Then all the following hold:*

(a) every hole in R intersects each of $X_1, \dots, X_k$ in exactly one vertex;
(b) every hole in R is of length k;
(c) for all $i \in \mathbb{Z}_k$, $R \setminus X_i$ is chordal.

A *hyperhole* is any graph H whose vertex set can be partitioned into $k \geq 4$ nonempty cliques, $X_0, \dots, X_{k-1}$ (with indices understood to be in $\mathbb{Z}_k$), such that for all $i \in \mathbb{Z}_k$, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$; under such circumstances, we also say that H is a hyperhole of *length* k, or that H is a *k-hyperhole*. Note that every k-hyperhole is a k-ring. When we say that "H is a k-hyperhole in G," or "G contains a k-hyperhole H," we

[14] Note that this implies that $X_0, \dots, X_{k-1}$ are cliques; that for all $i \in \mathbb{Z}_k$, X_i is anticomplete to $V(R) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$; and that $u_0^1, u_1^1, \dots, u_{k-1}^1, u_0^1$ is a k-hole in R.

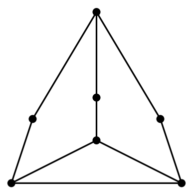


Figure 3.2: The 5-pyramid.

mean that H is a k-hyperhole that is an induced subgraph of G. The following is Corollary 1.3 from [21].

Theorem 3.8. *[21] Let $k \geq 4$ be an integer, and let R be a k-ring. Then* $\chi(R) = \max\Big(\{\omega(R)\} \cup \Big\{\Big\lceil \frac{|V(H)|}{\lfloor k/2 \rfloor} \Big\rceil \mid H \text{ is a } k\text{-hyperhole in } R\Big\}\Big)$.

It was further shown in [21] that the chromatic number of a ring can be computed in $O(n^3)$ time, and that an optimal coloring of a ring can be found in $O(n^6)$ time. In the present paper, we will repeatedly need to color rings (or their induced subgraphs), but the rings that we encounter are of a special type and can be colored faster (see Lemma 6.5; we remark that the algorithm from Lemma 6.5 relies on Theorem 3.8 above, but not on the coloring algorithms from [21]).

3.4 On the structure of $(4K_1, C_4, C_6, C_7)$-free graphs

In this subsection, we state the decomposition theorem for $(4K_1, C_4, C_6, C_7)$-free graphs from [23] (see Theorem 3.12 of the present paper). First, we need some definitions.

The *5-pyramid* is the graph on seven vertices represented in Figure 3.2.

A *5-basket* is a graph Q whose vertex set can be partitioned into sets $A, B_1, B_2, B_3, C_1, C_2, C_3, F$ such that all the following hold:

- $A, B_1, B_2, B_3, C_1, C_2, C_3$ are nonempty cliques;
- F is a (possibly empty) clique;
- cliques B_1, B_2, B_3 are pairwise anticomplete to each other;
- cliques C_1, C_2, C_3 are pairwise complete to each other;
- there exists an index $i^* \in \{1, 2, 3\}$ such that
  - A is complete to $(B_1 \cup B_2 \cup B_3) \setminus B_{i^*}$, and
  - A can be ordered as $A = \{a_1, \dots, a_t\}$ so that $N_Q(a_t) \cap B_{i^*} \subseteq \dots \subseteq N_Q(a_1) \cap B_{i^*} = B_{i^*}$;[15]
- A is anticomplete to $C_1 \cup C_2 \cup C_3$;
- for all indices $i \in \{1, 2, 3\}$, B_i is complete to C_i and anticomplete to $(C_1 \cup C_2 \cup C_3) \setminus C_i$;
- there exists an index $j^* \in \{1, 2, 3\}$ such that F is complete to $V(Q) \setminus (B_{j^*} \cup C_{j^*} \cup F)$ and anticomplete to $B_{j^*} \cup C_{j^*}$.

Under such circumstances, we say that $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ is a *5-basket partition* of the 5-basket Q.

Note that there are effectively two different types of 5-basket (depending on whether or not i^* and j^* are the same). These two types of 5-basket (up to a permutation of the index set $\{1, 2, 3\}$) are represented in Figure 3.3.

[15] Thus, a_1 is complete to $B_1 \cup B_2 \cup B_3$. Furthermore, B_{i^*} can be ordered as $B_{i^*} = \{b_1, \dots, b_p\}$ so that $a_1 \in N_Q(b_p) \cap A \subseteq \dots \subseteq N_Q(b_1) \cap A$.

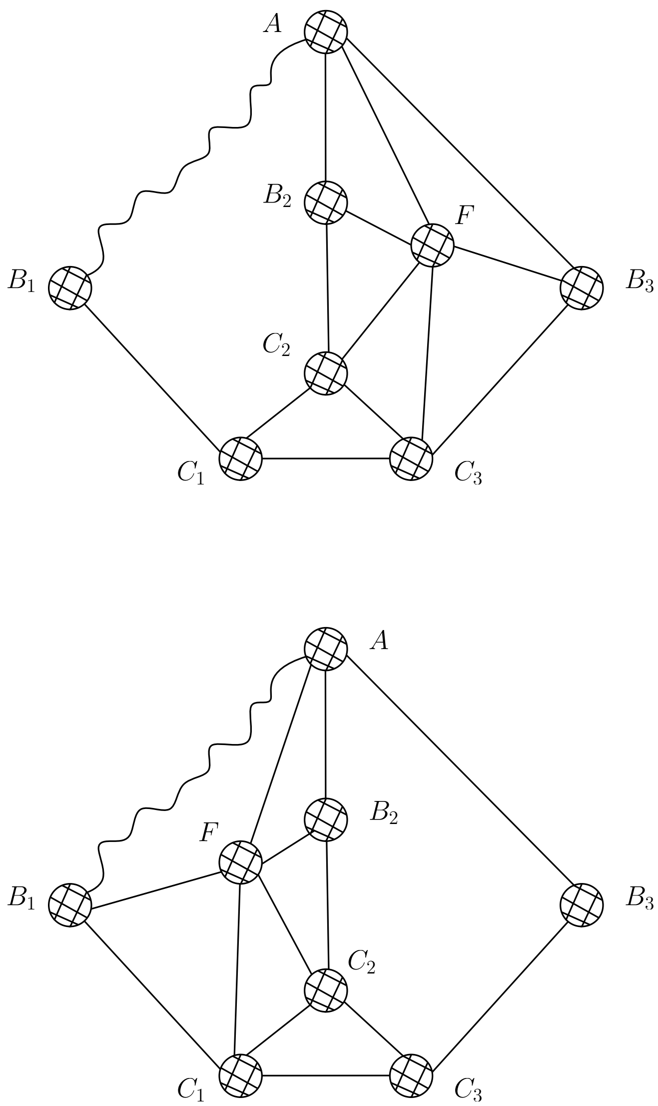


Figure 3.3: A 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$, and with $i^* = j^* = 1$ (top), or $i^* = 1$ and $j^* = 3$ (bottom).

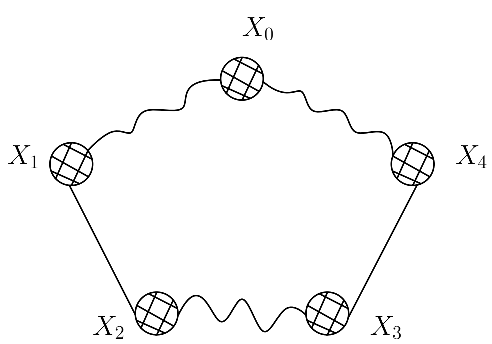


Figure 3.4: A 5-crown with 5-crown partition $(X_0, X_1, X_2, X_3, X_4)$ and $i^* = 0$.

A *5-crown* is a 5-ring R with ring partition $(X_0, X_1, X_2, X_3, X_4)$ such that for some index $i^* \in \mathbb{Z}_5$, we have that X_{i^*-1} is complete to X_{i^*-2}, and X_{i^*+1} is complete to X_{i^*+2}.[16] Under such circumstances, we say that $(X_0, X_1, X_2, X_3, X_4)$ is a *5-crown partition* of the 5-crown R. (See Figure 3.4.)

Lemma 3.9 (below) is an immediate consequence of Lemmas 2.1, 2.5 and 2.11 from [23].

Lemma 3.9. *[23] 5-Baskets and 5-crowns are anticonnected, contain no simplicial vertices, and are $(4K_1, C_4, C_6, C_7)$-free.*

Theorems 3.10, 3.11, and 3.12 (below) are, respectively, Theorems 2.3, 2.4, and 2.2 from [23].

Theorem 3.10. *[23] Let G be a graph. Then the following are equivalent:*

- *G is a $(4K_1, C_4, C_6, C_7)$-free graph that contains an induced 5-pyramid and does not contain a simplicial vertex;*
- *G has exactly one nontrivial anticomponent, and this anticomponent is a 5-basket.*

Theorem 3.11. *[23] Let G be a graph. Then the following are equivalent:*

- *G is a $(4K_1, C_4, C_6, C_7, 5\text{-pyramid})$-free graph that does not contain a simplicial vertex;*
- *G has exactly one nontrivial anticomponent, and this anticomponent is a 5-crown.*

Theorem 3.12. *[23] Let G be a graph. Then the following two statements are equivalent:*

- *G is a $(4K_1, C_4, C_6, C_7)$-free graph that does not contain a simplicial vertex;*
- *G has exactly one nontrivial anticomponent, and this anticomponent is either a 5-basket or a 5-crown.*

3.5 $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7

A *special partition* of a graph G is a partition $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$ of $V(G)$ into (possibly empty) cliques such that all the following are satisfied:[17]

(a) cliques $X_0, \dots, X_6$ are all nonempty;[18]
(b) for all $i \in \mathbb{Z}_7$, X_i is complete to X_{i-1}, X_{i+1} and anticomplete to $X_{i+2}, X_{i+3}, X_{i+4}, X_{i+5}$;
(c) for all $i \in \mathbb{Z}_7$, X_i is complete to $Y_i, Y_{i+3}, Y_{i+6}, Z_i, Z_{i+3}, Z_{i+4}, Z_{i+5}, Z_{i+6}, W$ and anticomplete to $Y_{i+1}, Y_{i+2}, Y_{i+4}, Y_{i+5}, Z_{i+1}, Z_{i+2}$;
(d) for all $i \in \mathbb{Z}_7$, if $Y_i \neq \emptyset$, then $Y_{i+1}, Y_{i+2}, Y_{i+5}, Y_{i+6}, Z_{i+5}, Z_{i+6}$ are all empty, and at most one of Y_{i+3}, Y_{i+4} is nonempty;[19]
(e) for all $i \in \mathbb{Z}_7$, if $Z_i \neq \emptyset$, then Z_{i+2}, Z_{i+5} are empty;[20]
(f) for all $i \in \mathbb{Z}_7$, Y_i is complete to $Y_{i+3}, Y_{i+4}, Z_i, Z_{i+1}, Z_{i+3}, Z_{i+4}, W$ and anticomplete to Z_{i+2};
(g) for all $i \in \mathbb{Z}_7$, Z_i is complete to $Z_{i+1}, Z_{i+3}, Z_{i+4}, Z_{i+6}, W$.

Note that if $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$ is a special partition of a graph G, then by definition, the following hold:

[16] Rings were defined in subsection 3.3. To help the reader, here is an equivalent definition of a 5-crown that makes no reference to rings: a *5-crown* is a graph R whose vertex set can be partitioned into five nonempty sets, say X_0, X_1, X_2, X_3, X_4 (with indices understood to be in $\mathbb{Z}_5$), such that the following two conditions are satisfied:

- for all $i \in \mathbb{Z}_5$, X_i can be ordered as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$;
- for some index $i^* \in \mathbb{Z}_5$, we have that X_{i^*-1} is complete to X_{i^*-2}, and X_{i^*+1} is complete to X_{i^*+2}.

Note that the first bullet point above implies that X_0, X_1, X_2, X_3, X_4 are all cliques.

[17] Indices are understood to be in $\mathbb{Z}_7$.

[18] Cliques $Y_0, \dots, Y_6, Z_0, \dots, Z_6, W$ may possibly be empty.

[19] In particular, at most two of the cliques $Y_0, \dots, Y_6$ are nonempty.

[20] In particular, at most three of the cliques $Z_0, \dots, Z_6$ are nonempty.

- for all $i \in \mathbb{Z}_7$, Y_i is complete to $X_i \cup X_{i+1} \cup X_{i+4}$ and anticomplete to $X_{i+2} \cup X_{i+3} \cup X_{i+5} \cup X_{i+6}$;
- for all $i \in \mathbb{Z}_7$, Z_i is complete to $X_i \cup X_{i+1} \cup X_{i+2} \cup X_{i+3} \cup X_{i+4}$ and anticomplete to $X_{i+5} \cup X_{i+6}$.

The following was proven in section 3 of [14].[21]

Theorem 3.13. *[14] Let G be a $(4K_1, C_4, C_6)$-free graph that contains an induced C_7. Then G admits a special partition.*

We remark that the converse of Theorem 3.13 also holds: if a graph admits a special partition, then it is $(4K_1, C_4, C_6)$-free and contains an induced C_7. This follows by routine checking, but it was not proven (indeed, it was not even stated) in [14]. We prove this in Lemma 7.3.

For a positive integer k, a *k-uniform partition* of a graph G is a partition of $V(G)$ into k cliques, any two of which are either complete or anticomplete to each other. The following is Theorem 2.6 from [14] (this result is an immediate corollary of Theorem 3.13 above).

Theorem 3.14. *[14] Let G be a $(4K_1, C_4, C_6)$-free graph that contains an induced C_7. Then G admits a k-uniform partition, with $7 \leq k \leq 13$.*

Note that any two vertices that belong to the same clique of a k-uniform partition of a graph G, are true twins in G. So, if G admits a k-uniform partition, but does not contain a pair of true twins, then G has at most k vertices. In particular, Theorem 3.14 implies that any $(4K_1, C_4, C_6)$-free graph that contains an induced C_7, and does not contain a pair of true twins, has at most 13 vertices.

4 Recognizing $(4K_1, C_4, C_6, C_7)$-free graphs

Clearly, $(4K_1, C_4, C_6, C_7)$-free graphs can be recognized in $O(n^7)$ time. However, by using Theorem 3.12, we can recognize graphs in this class in only $O(n^3)$ time (see Theorem 4.4). In particular, our recognition algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs does not directly search for the four forbidden induced subgraphs; instead, it checks whether the input graph has the structure described in Theorem 3.12.[22] Thus, it is possible that our algorithm (correctly) determines that the input graph is not $(4K_1, C_4, C_6, C_7)$-free without actually finding any of the four forbidden induced subgraphs in the input graph.

Our first goal is to give $O(n^2)$ time recognition algorithms for 5-crowns and 5-baskets (see Lemmas 4.1 and 4.3, respectively).

Lemma 4.1. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *Either the true statement that G is a 5-crown, together with a 5-crown partition of G, or the true statement that G is not a 5-crown;*
- ***Running time:*** *$O(n^2)$.*

[21] We remark that the term “special partition” was never used in [14]; we use that term as a convenient shorthand. The reader can easily check that the structure described in section 3 of [14] is precisely what we named a “special partition,” with one exception. Item (d) from our definition of a special partition states that at most one of Y_{i+3}, Y_{i+4} is nonempty. The reader may have noticed that, in [14], it is stated that exactly one of Y_{i+3}, Y_{i+4} is nonempty. However, this is quite obviously a typo: it is possible that both Y_{i+3}, Y_{i+4} are empty, and in fact, it may well be that $Y_0, \dots, Y_6, Z_0, \dots, Z_6, W$ are all empty (consider the case when G is a 7-hyperhole).

[22] Here, simplicial vertices pose a bit of a problem because deleting a simplicial vertex may possibly eliminate an induced $4K_1$; see the proof of Theorem 4.4 for a way to deal with this.

Proof. We first call the $O(n^2)$ time algorithm from Lemma 3.6 with input G. If the algorithm returns the answer that G is not a ring, or that G is a ring of length other than five, then we return the answer that G is not a 5-crown, and we stop. Assume now that the algorithm returned the answer that G is a 5-ring, together with a ring partition $(X_0, \dots, X_4)$ of G. Then we check if there exists an index $i^* \in \mathbb{Z}_5$ such that X_{i^*-1} is complete to X_{i^*-2}, and X_{i^*+1} is complete to X_{i^*+2} in G; clearly, this can be done in $O(n^2)$ time by examining adjacency between consecutive X_i's. If we found such an index i^*, then we return the answer that G is a 5-crown with 5-crown partition $(X_0, \dots, X_4)$, and we stop. Otherwise, we return the answer that G is not a 5-crown, and we stop. Clearly, the algorithm is correct, and its running time is $O(n^2)$. $\square$

We now turn to 5-baskets. We first state and prove a technical lemma (see Lemma 4.2 below), which we then use to construct our $O(n^2)$ time recognition algorithm for 5-baskets (see Lemma 4.3).

Lemma 4.2. *Let Q be a 5-basket that contains no pair of true twins, and assume that $|V(Q)| \geq 12$. Then there exists a unique vertex $a \in V(Q)$ such that $d_{\overline{Q}}(a) = 3$.*[23] *Further, let c_1, c_2, c_3 be the three nonneighbors of a in Q (listed in any order), and define sets $A, B_1, B_2, B_3, C_1, C_2, C_3, F$ as follows:*

- *A is the set of all vertices of $V(Q) \setminus \{c_1, c_2, c_3\}$ that are anticomplete to $\{c_1, c_2, c_3\}$;*[24]
- *for all $i \in \{1, 2, 3\}$, B_i is the set of all vertices in $V(Q) \setminus \{a, c_1, c_2, c_3\}$ that are complete to $\{a, c_i\}$ and anticomplete to $\{c_1, c_2, c_3\} \setminus \{c_i\}$;*
- *for all $i \in \{1, 2, 3\}$, $C_i := \{c_i\}$;*
- *$F := V(Q) \setminus \Big(A \cup B_1 \cup B_2 \cup B_3 \cup C_1 \cup C_2 \cup C_3\Big)$.*

Then $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ is a 5-basket partition of Q.

Proof. Let $(A'; B_1', B_2', B_3'; C_1', C_2', C_3'; F')$ be a 5-basket partition of the 5-basket Q. Let indices i^* and j^* be as in the definition of a 5-basket; by symmetry, we may assume that either $i^* = j^* = 1$, or $i^* = 1$ and $j^* = 3$ (see Figure 3.3). Note that for all $X \in \{B_2', B_3', C_1', C_2', C_3', F'\}$, any two vertices of X are true twins in Q; since Q contains no pair of true twins, and since $B_2', B_3', C_1', C_2', C_3'$ are all nonempty, we have that $B_2', B_3', C_1', C_2', C_3'$ are all singletons, and that $|F'| \leq 1$. Since $|V(Q)| \geq 12$, it follows that $|A' \cup B_1'| \geq 6$.

Now, let $A' = \{a_1, \dots, a_t\}$ be an ordering of A' such that $N_Q(a_t) \cap B_1' \subseteq \dots \subseteq N_Q(a_1) \cap B_1' = B_1'$, and let $B_1' = \{b_1, \dots, b_p\}$ be an ordering of B_1' such that $a_1 \in N_Q(b_p) \cap A' \subseteq \dots \subseteq N_Q(b_1) \cap A'$, as in the definition of a 5-basket. Note that $N_Q[a_1] \setminus B_1' = \dots = N_Q[a_t] \setminus B_1'$ and $N_Q[b_1] \setminus A' = \dots = N_Q[b_p] \setminus A'$. Since Q contains no pair of true twins, we now deduce that $N_Q(a_t) \cap B_1' \subsetneqq \dots \subsetneqq N_Q(a_1) \cap B_1' = B_1'$ and $a_1 \in N_Q(b_p) \cap A' \subsetneqq \dots \subsetneqq N_Q(b_1) \cap A'$; consequently, $|B_1'| \leq |A'| \leq |B_1'| + 1$. Since $|A' \cup B_1'| \geq 6$, it follows that $|A'| \geq 3$ and $|B_1'| \geq 3$.

Claim 1. Vertex a_1 is the unique vertex of Q that has exactly three nonneighbors in Q.

Proof of Claim 1. First, it follows from the definition of a 5-basket that $V(Q) \setminus N_Q[a_1] = C_1' \cup C_2' \cup C_3'$; since C_1', C_2', C_3' are pairwise disjoint singletons, it follows that $d_{\overline{Q}}(a_1) = 3$. On the other hand, for all $i \in \{2, \dots, t\}$, we have that $d_Q(a_i) < d_Q(a_1)$, and consequently, $d_{\overline{Q}}(a_i) > d_{\overline{Q}}(a_1) = 3$. Further, for all $i \in \{1, 2, 3\}$, we have that B_i' is anticomplete to $(B_1' \cup B_2' \cup B_3' \cup C_1' \cup C_2' \cup C_3') \setminus (B_i' \cup C_i')$; since sets $B_1', B_2', B_3', C_1', C_2', C_3'$ are all nonempty and pairwise disjoint, it follows that each vertex in $B_1' \cup B_2' \cup B_3'$ has at least four nonneighbors in Q. Next, for all $i \in \{1, 2, 3\}$, we have that C_i' is anticomplete to $(A' \cup B_1' \cup B_2' \cup B_3') \setminus B_i'$; since sets

[23] That is, Q has exactly one vertex (called a) that has exactly three nonneighbors in Q.

[24] Note that this means that $a \in A$.

A', B_1', B_2', B_3' are all nonempty and pairwise disjoint, and since $|A'| \geq 3$, it follows that each vertex in $C_1' \cup C_2' \cup C_3'$ has at least five nonneighbors in Q. It remains to show that no vertex in F' has exactly three nonneighbors in Q. We may assume that $F' \neq \emptyset$, for otherwise, there is nothing to show. Now $|F'| = 1$, and we set $F' = \{f\}$. If $j^* = 1$, then $V(Q) \setminus N_Q[f] = B_1' \cup C_1'$, and it follows that $d_{\overline{Q}}(f) \geq 4$.[25] On the other hand, if $j^* = 3$, then $V(Q) \setminus N_Q[f] = B_3' \cup C_3'$, and it follows that $d_{\overline{Q}}(f) = 2$.[26] This proves Claim 1. ♦

Set $a := a_1$; by Claim 1, a is the only vertex of Q that has exactly three nonneighbors in Q. Let c_1, c_2, c_3 be the three nonneighbors of a (listed in any order). Since $V(Q) \setminus N_G[a] = C_1' \cup C_2' \cup C_3'$, and since C_1', C_2', C_3' are all singletons, we see that there exists some permutation σ of the index set $\{1, 2, 3\}$ such that $C'_{\sigma(1)} = \{c_1\}$, $C'_{\sigma(2)} = \{c_2\}$, and $C'_{\sigma(3)} = \{c_3\}$. But now if sets $A, B_1, B_2, B_3, C_1, C_2, C_3, F$ are defined as in the statement of the lemma, then all the following hold: $A = A'$; $C_i = C'_{\sigma(i)}$ and $B_i = B'_{\sigma(i)}$ for all $i \in \{1, 2, 3\}$; $F = F'$. Since $(A'; B_1', B_2', B_3'; C_1', C_2', C_3'; F')$ is a 5-basket partition of the 5-basket Q, so is the partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$. This completes the argument. □

We remind the reader that for a set S of sets, we write $\bigcup S := \bigcup_{X \in S} X$.

Lemma 4.3. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *One of the following:*
  - *The true statement that G is a 5-basket, together with a 5-basket partition of G,*
  - *The true statement that G is not a 5-basket;*
- ***Running time:*** *$O(n^2)$.*

Proof. We first call the $O(n^2)$ time algorithm from Lemma 3.5 with input G, and we obtain the partition $\mathcal{P}$ of $V(G)$ into true twin classes of G, as well as the quotient graph $Q := G_{\mathcal{P}}$. Obviously, Q contains no pair of true twins.

> **Claim 1.** G is a 5-basket if and only if Q is a 5-basket. Moreover, if Q is a 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$, then G is a 5-basket with 5-basket partition $(\bigcup A; \bigcup B_1, \bigcup B_2, \bigcup B_3; \bigcup C_1, \bigcup C_2, \bigcup C_3; \bigcup F)$.

Proof of Claim 1. It is obvious that if Q is a 5-basket with an associated 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$, then G is a 5-basket with an associated 5-basket partition $(\bigcup A; \bigcup B_1, \bigcup B_2, \bigcup B_3; \bigcup C_1, \bigcup C_2, \bigcup C_3; \bigcup F)$. It remains to show that if G is a 5-basket, then so is Q. But this readily follows from the fact that every true twin class of a 5-basket is included (as a subset) in one of the eight sets of an (any) associated 5-basket partition. This proves Claim 1. ♦

In view of Claim 1, it remains to check whether Q is a 5-basket, and if so, to find an associated 5-basket partition of Q. If $|V(Q)| \leq 11$, then this can be done by brute force in $O(1)$ time. So, assume that $|V(Q)| \geq 12$. We then form the graph $\overline{Q}$, and we compute the degrees of all vertices in this graph; this can be done in $O(n^2)$ time. If $\overline{Q}$ contains no vertices of degree three, or if it contains more than one such vertex, then Lemma 4.2 guarantees that Q is not a 5-basket. So, suppose that $\overline{Q}$ has exactly one vertex (call it a) of degree three. Let c_1, c_2, c_3 be the three nonneighbors of a in Q (listed in any order), and let sets $A, B_1, B_2, B_3, C_1, C_2, C_3, F$ be as in the statement of Lemma 4.2, i.e. let these sets be defined as follows:

- A is the set of all vertices of $V(Q) \setminus \{c_1, c_2, c_3\}$ that are anticomplete to $\{c_1, c_2, c_3\}$;[27]

[25] We are using the fact that $B_1' \cap C_1' = \emptyset$, $|B_1'| \geq 3$, and $|C_1'| = 1$.

[26] We are using the fact that B_3' and C_3' are disjoint singletons.

[27] Note that this means that $a \in A$.

- for all $i \in \{1,2,3\}$, B_i is the set of all vertices in $V(Q) \setminus \{a, c_1, c_2, c_3\}$ that are complete to $\{a, c_i\}$ and anticomplete to $\{c_1, c_2, c_3\} \setminus \{c_i\}$;
- for all $i \in \{1,2,3\}$, $C_i := \{c_i\}$;
- $F := V(Q) \setminus \Big(A \cup B_1 \cup B_2 \cup B_3 \cup C_1 \cup C_2 \cup C_3\Big)$.

Clearly, these eight sets can be computed in $O(n)$ time.

By Lemma 4.2, either $P := (A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ is a 5-basket partition of Q, or Q is not a 5-basket. We check if P is a 5-basket partition of Q, as follows. We first check in $O(n^2)$ time if all the following hold:

(1) $A, B_1, B_2, B_3, C_1, C_2, C_3$ are all nonempty cliques;
(2) F is a (possibly empty) clique;
(3) B_1, B_2, B_3 are anticomplete to each other;
(4) C_1, C_2, C_3 are complete to each other;
(5) A is anticomplete to C_1, C_2, C_3;
(6) for all $i \in \{1,2,3\}$, B_i is complete to C_i and anticomplete to $(C_1 \cup C_2 \cup C_3) \setminus C_i$;
(7) there exists an index $j \in \{1,2,3\}$ such that F is complete to $V(Q) \setminus (B_j \cup C_j)$ and anticomplete to $B_j \cup C_j$.

If any one of (1)-(7) fails, then P is not a 5-basket partition of Q. Suppose now that (1)-(7) all hold. We then compute the degrees (in Q) of all vertices in A, and we order A as $A = \{a_1, \dots, a_t\}$ so that $d_Q(a_t) \leq \dots \leq d_Q(a_1)$; this can be done in $O(n^2)$ time. We now check in further $O(n^2)$ time if $N_Q[a_t] \subseteq \dots \subseteq N_Q[a_1]$; if not, then P is not a 5-basket partition of Q. Assume now that $N_Q[a_t] \subseteq \dots \subseteq N_Q[a_1]$. We then check in $O(n)$ time whether there exists an index $i \in \{1,2,3\}$ such that a_t is complete to $(B_1 \cup B_2 \cup B_3) \setminus B_i$; if such an index i does not exist, then P is not a 5-basket partition of Q. So assume we have found such an index i; it then follows from our ordering of A that A is complete to $(B_1 \cup B_2 \cup B_3) \setminus B_i$. We now check in $O(n)$ time if a_1 is complete to B_i. If a_1 is not complete to B_i, then P is not a 5-basket partition of Q. On the other hand, if a_1 is complete to B_i, then P is indeed a 5-basket partition of Q.

We have now determined whether Q is a 5-basket, and if so, we have found an associated 5-basket partition of Q. If Q is not a 5-basket, then we return the answer that G is not a 5-basket, and we stop. On the other hand, if we have determined that Q is a 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$, then we return the answer that G is a 5-basket with 5-basket partition $(\bigcup A; \bigcup B_1, \bigcup B_2, \bigcup B_3; \bigcup C_1, \bigcup C_2, \bigcup C_3; \bigcup F)$, and we stop. By Claim 1, this is correct.

Clearly, the algorithm is correct, and its running time is $O(n^2)$. □

We are now ready to describe our $O(n^3)$ time recognition algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs.

Theorem 4.4. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *Either the true statement that G is $(4K_1, C_4, C_6, C_7)$-free, or the true statement that G is not $(4K_1, C_4, C_6, C_7)$-free;*
- ***Running time:*** *$O(n^3)$.*

Proof. We first call the $O(n^3)$ time algorithm from Lemma 3.3 with input G, and we obtain a maximal sequence $v_1, \dots, v_s$ ($s \geq 0$) of pairwise distinct vertices of G such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$. Suppose first that $V(G) = \{v_1, \dots, v_s\}$. Then $v_1, \dots, v_s$ is a simplicial elimination ordering of G, and so Theorem 3.2 guarantees that G is chordal, and in particular, that G is (C_4, C_6, C_7)-free. Further, we can compute $\alpha(G)$ in $O(n^2)$ time by calling the algorithm from Lemma 3.4 with input G and $v_1, \dots, v_s$. If $\alpha(G) \geq 4$, then G is not $4K_1$-free; in this case, we return the answer that G is not $(4K_1, C_4, C_6, C_7)$-free, and we stop. On the other hand, if $\alpha(G) \leq 3$, then G is $4K_1$-free; in this case, we return the answer that G is $(4K_1, C_4, C_6, C_7)$-free, and we stop.

From now on, we assume that $\{v_1, \dots, v_s\} \subsetneqq V(G)$. We then form the graph $H := G \setminus \{v_1, \dots, v_s\}$ and the set U of all universal vertices of H; this takes $O(n^2)$ time. By the maximality of $v_1, \dots, v_s$, the graph H has no simplicial vertices, and in particular, H is not complete; thus, $U \subsetneqq V(H)$. We now form the graph $Q := H \setminus U$ in further $O(n^2)$ time.

Claim 1. All holes of G are in fact holes of Q. Consequently, for all $X \subsetneqq V(Q)$, if $Q \setminus X$ is chordal, then $G \setminus X$ is also chordal. Furthermore, if Q is (C_4, C_6, C_7)-free, then so is G.

Proof of Claim 1. The first statement follows from the fact that holes contain no simplicial and no universal vertices. The second and third statement follow from the first. This proves Claim 1. ♦

Using the $O(n^2)$ time algorithms from Lemmas 4.1 and 4.3, we check if Q is a 5-crown or a 5-basket (or neither). If Q is neither a 5-crown nor a 5-basket, then we return the answer that G is not $(4K_1, C_4, C_6, C_7)$-free, and we stop; Theorem 3.12 guarantees that this is correct. From now on, we assume that Q is either a 5-basket or a 5-crown, and that we have also obtained a relevant partition of Q, as specified in the algorithms from Lemmas 4.1 and 4.3.

By Lemma 3.9, Q is $(4K_1, C_4, C_6, C_7)$-free, and so by Claim 1, G is (C_4, C_6, C_7)-free. It remains to check if G is $4K_1$-free. Clearly, this could be done in $O(n^4)$ time by brute force; however, this is too slow for our purposes, and so instead, we will determine whether G is $4K_1$-free by computing the stability number of several (at most four) chordal induced subgraphs of G, using the algorithm from Lemma 3.4. We consider two cases: when Q is a 5-crown, and when Q is a 5-basket.

Case 1: Q is a 5-crown. Let $(X_0, X_1, X_2, X_3, X_4)$ be a 5-crown partition of Q, returned by the algorithm from Lemma 4.1. By the definition of a 5-crown, there exists an index $i \in \mathbb{Z}_5$ such that X_i is complete to X_{i+1}, and clearly, such an index i can be found in $O(n^2)$ time. By symmetry, we may assume that $i = 1$, so that X_1 is complete to X_2. Then no stable set of G intersects both X_1 and X_2; consequently, every stable set of G is in fact a stable set of $G \setminus X_1$ or $G \setminus X_2$. Thus, $\alpha(G) = \max\{\alpha(G \setminus X_1), \alpha(G \setminus X_2)\}$. Further, by Lemma 3.7(c), both $Q \setminus X_1$ and $Q \setminus X_2$ are chordal; Claim 1 now implies that $G \setminus X_1$ and $G \setminus X_2$ are both chordal, and therefore (by Theorem 3.2) admit a simplicial elimination ordering. We can find a simplicial elimination ordering of $G \setminus X_1$ in $O(n^3)$ time using the algorithm from Lemma 3.3, and then we can find $\alpha(G \setminus X_1)$ in further $O(n^2)$ time using the algorithm from Lemma 3.4. Similarly, we can find $\alpha(G \setminus X_2)$ in $O(n^3)$ time using the algorithms from Lemmas 3.3 and 3.4. Finally, we compute $\alpha(G) = \max\{\alpha(G \setminus X_1), \alpha(G \setminus X_2)\}$. If $\alpha(G) \geq 4$, then G is not $4K_1$-free; in this case, we return the answer that G is not $(4K_1, C_4, C_6, C_7)$-free, and we stop. Otherwise, G is $4K_1$-free; in this case, we return the answer that G is $(4K_1, C_4, C_6, C_7)$-free,[28] and we stop.

Case 2: Q is a 5-basket. Let $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ be a 5-basket partition of Q, returned by the algorithm from Lemma 4.3.

Claim 2. For every hole in G, there exists some index $i \in \{1, 2, 3\}$ such that the hole intersects both B_i and C_i.

Proof of Claim 2. By Claim 1, every hole of G is a hole of Q. Further, by Lemma 3.9, Q is $(4K_1, C_4, C_6, C_7)$-free; consequently, every hole in Q is of length five. Thus, it suffices to show that for every 5-hole in Q, there exists some index $i \in \{1, 2, 3\}$ such that the 5-hole intersects both B_i and C_i. Let $y_0, \dots, y_4, y_0$ (with indices in $\mathbb{Z}_5$) be a 5-hole in Q, and set $Y := \{y_0, \dots, y_4\}$.

[28]We have already checked that G is (C_4, C_6, C_7)-free, and so this is correct.

First, we observe that for all $X \in \{A, B_1, B_2, B_3, C_1, C_2, C_3, F\}$, and all distinct vertices $x, x' \in X$, one of x, x' dominates the other in Q. Since no vertex of a hole dominates any other vertex of that hole, we deduce that none of $A, B_1, B_2, B_3, C_1, C_2, C_3, F$ contains more than one vertex of Y.

Next, we can obtain a simplicial elimination ordering of $Q \setminus A$ as follows: we first list all vertices of $B_1 \cup B_2 \cup B_3$ (in any order), then we list all vertices of F (in any order), and finally, we list all vertices of $C_1 \cup C_2 \cup C_3$ (in any order). It now follows from Theorem 3.2 that $Q \setminus A$ is chordal, and consequently, $Y \cap A \neq \emptyset$. We have already shown that $|Y \cap A| \leq 1$, and we now deduce that $|Y \cap A| = 1$. By symmetry, we may assume that $Y \cap A = \{y_0\}$. Since y_0 is complete to $\{y_1, y_4\}$, and since A is anticomplete to $C_1 \cup C_2 \cup C_3$, we now have that $y_1, y_4 \in B_1 \cup B_2 \cup B_3 \cup F$. Since $|Y \cap F| \leq 1$, we deduce that at least one of y_1, y_4 belongs to $B_1 \cup B_2 \cup B_3$; by symmetry, we may assume that $y_1 \in B_1$. Since $|Y \cap B_1| \leq 1$, we deduce that $Y \cap B_1 = \{y_1\}$. Since y_1 is adjacent to y_2, and since B_1 is anticomplete to $B_2 \cup B_3 \cup C_2 \cup C_3$, we now have that $y_2 \in A \cup B_1 \cup C_1 \cup F$. But since $Y \cap A = \{y_0\}$ and $Y \cap B_1 = \{y_1\}$, we in fact have that $y_2 \in C_1 \cup F$. If $y_2 \in F$, then $\{y_0, y_1, y_2\}$ is a triangle (*i.e.* a clique of size three),[29] contrary to the fact that holes contain no triangles. So, $y_2 \in C_1$. But now Y intersects both B_1 and C_1. This proves Claim 2. ♦

Let $C := C_1 \cup C_2 \cup C_3$, and for all $i \in \{1, 2, 3\}$, let $D_i := B_i \cup (C \setminus C_i)$. Further, set $G_C := G \setminus C$, and for all $i \in \{1, 2, 3\}$, set $G_i := G \setminus D_i$. Forming sets C, D_1, D_2, D_3 and graphs G_C, G_1, G_2, G_3 takes $O(n^2)$ time. By Claim 2, graphs G_C, G_1, G_2, G_3 are all chordal, and consequently (by Theorem 3.2), they all admit a simplicial elimination ordering. We find a simplicial elimination ordering for each of these four chordal graphs using the $O(n^3)$ time algorithm from Lemma 3.3, and then we compute their stability number in further $O(n^2)$ time using the algorithm from Lemma 3.4.

Claim 3. $\alpha(G) = \max\{\alpha(G_C), \alpha(G_1), \alpha(G_2), \alpha(G_3)\}$.

Proof of Claim 3. Clearly, it is enough to show that every stable set of G is in fact a stable set of one of G_C, G_1, G_2, G_3. For this, we need only show that every stable set of G has an empty intersection with at least one of C, D_1, D_2, D_3. Let S be a stable set of G. We may assume that $S \cap C \neq \emptyset$, for otherwise, we are done. Fix $i \in \{1, 2, 3\}$ such that $S \cap C_i \neq \emptyset$. Since C_i is complete to D_i, and since S is a stable set, we have that $S \cap D_i = \emptyset$. This proves Claim 3. ♦

We have already computed $\alpha(G_C), \alpha(G_1), \alpha(G_2), \alpha(G_3)$; in view of Claim 3, we can now compute $\alpha(G)$ simply by taking the maximum of these four numbers. If $\alpha(G) \geq 4$, then G is not $4K_1$-free; in this case, we return the answer that G is not $(4K_1, C_4, C_6, C_7)$-free, and we stop. Otherwise, G is $4K_1$-free; in this case, we return the answer that G is $(4K_1, C_4, C_6, C_7)$-free,[30] and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. □

5 Cliques and clique covers of $(4K_1, C_4, C_6, C_7)$-free graphs

Theorem 5.1. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *Either a list of all maximal cliques of G, or the true statement that G is not $(4K_1, C_4, C_6, C_7)$-free;*
- ***Running time:*** *$O(n^3)$.*

[29] Indeed, y_0y_1 and y_1y_2 are edges because $y_0, \dots, y_4, y_0$ is a hole, and y_2y_0 is an edge because $y_2 \in F$, $y_0 \in A$, and F is complete to A.

[30] We have already checked that G is (C_4, C_6, C_7)-free, and so this is correct.

Furthermore, if the algorithm returns all maximal cliques of the input graph G, then the number of cliques returned is at most n, where n is the number of vertices of G.

Proof. First, it is clear that, for a clique C of G, we can check in $O(n^2)$ time whether C is a maximal clique of G (we simply check whether some vertex of $V(G) \setminus C$ is complete to C). The algorithm proceeds as follows. First, we compute a list of at most n cliques of G such that all maximal cliques of G are on this list (but not all cliques on the list need be maximal). Then, we check in $O(n^3)$ time which of those cliques are maximal, and we return those maximal cliques.

We now turn to the technical details. We begin by finding a maximal sequence $v_1, \dots, v_s$ ($s \geq 0$) of vertices of G such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$; this can be done in $O(n^3)$ time by calling the algorithm from Lemma 3.3 with input G. For each $i \in \{1, \dots, v_i\}$, we form the set $D_i = N_G[v_i] \setminus \{v_1, \dots, v_{i-1}\}$. Clearly, sets $D_1, \dots, D_s$ can be computed in $O(n^2)$ time, and they are all cliques.

Claim 1. For every maximal clique C of G, if $C \cap \{v_1, \dots, v_s\} \neq \emptyset$, then there exists an index $i \in \{1, \dots, s\}$ such that $C = D_i$.

Proof of Claim 1. Fix a maximal clique C of G, and assume that $C \cap \{v_1, \dots, v_s\} \neq \emptyset$. Let $i \in \{1, \dots, s\}$ be minimal with the property that $v_i \in C$. Then $C \subseteq N_G[v_i] \setminus \{v_1, \dots, v_{i-1}\} = D_i$. Since both C and D_i are cliques, the maximality of C guarantees that $C = D_i$. This proves Claim 1. ♦

Suppose first that $V(G) = \{v_1, \dots, v_s\}$. Then $s = n$, and by Claim 1, all maximal cliques of G are on the list $D_1, \dots, D_s$. We check in $O(n^3)$ time which of the cliques $D_1, \dots, D_s$ are maximal in G, we return those maximal cliques, and we stop.

From now on, we assume that $\{v_1, \dots, v_s\} \subsetneqq V(G)$. We now form the graph $H := G \setminus \{v_1, \dots, v_s\}$, as well as the set U of all universal vertices of H; this takes $O(n^2)$ time. By the maximality of $v_1, \dots, v_s$, we know that H contains no simplicial vertices, and consequently, H is not complete; thus, $U \subsetneqq V(H)$. We now form the graph $Q := H \setminus U$ in further $O(n^2)$ time.

Claim 2. For every maximal clique C of G, either

- there exists an index $i \in \{1, \dots, s\}$ such that $C = D_i$, or
- there exists some maximal clique C' of the graph Q such that $C = C' \cup U$.

Proof of Claim 2. Fix a maximal clique C of G. We may assume that $C \cap \{v_1, \dots, v_s\} = \emptyset$, for otherwise, we are done by Claim 1. But then C is a maximal clique of H. Since U is the set of all universal vertices of H, it is clear that $U \subseteq C$, and that $C' := C \setminus U$ is a maximal clique of $H \setminus U = Q$. This proves Claim 2. ♦

By Theorem 3.12, if G is $(4K_1, C_4, C_6, C_7)$-free, then Q is a 5-crown or a 5-basket. We can check if Q is a 5-crown or a 5-basket (or neither) using the $O(n^2)$ time algorithms from Lemmas 4.1 and 4.3. If Q is neither a 5-crown nor a 5-basket, then we return the answer that G is not $(4K_1, C_4, C_6, C_7)$-free, and we stop. From now on, we may assume that we have determined that Q is either a 5-crown or a 5-basket, and that we have also obtained a relevant partition of Q, as specified in the algorithms from Lemmas 4.1 and 4.3.

In view of Claim 2, our goal is now to compute all maximal cliques of Q. We consider two cases: when Q is a 5-crown, and when Q is a 5-basket.

Case 1: Q is a 5-crown. Let $(X_0, \dots, X_4)$ be a 5-crown partition of the 5-crown Q, returned by the algorithm from Lemma 4.1. We now compute the degrees (in Q) of all vertices of Q, and then for each $i \in \mathbb{Z}_5$, we order X_i as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $d_Q(u_i^{|X_i|}) \leq \dots \leq d_Q(u_i^1)$; this can be done in $O(n^2)$ time. Since we already know that $(X_0, \dots, X_4)$ is a 5-crown partition of Q, we see that for all $i \in \mathbb{Z}_5$, we have that $X_i \subseteq N_Q[u_i^{|X_i|}] \subseteq \dots \subseteq N_Q[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$. Now, for all $i \in \mathbb{Z}_5$, and all $j \in \{1, \dots, |X_i|\}$, we compute the set $C_i^j := \{u_i^1, \dots, u_i^j\} \cup (N_Q(u_i^j) \cap X_{i+1})$;

the family $\{C_i^j\}$ can be computed in $O(n^2)$ time. It readily follows from our orderings of the sets X_i that each set C_i^j is a clique, and furthermore, that every maximal clique of Q is of the form C_i^j.[31] It then follows from Claim 2 that all maximal cliques of G are on the list $D_1, \dots, D_s, C_0^1 \cup U, \dots, C_0^{|X_0|} \cup U, \dots, C_4^1 \cup U, \dots, C_4^{|X_4|} \cup U$; the number of cliques on this list is $s + |V(Q)| \leq n$. We now check in $O(n^3)$ time which cliques on our list are maximal in G, we return those maximal cliques, and we stop.

Case 2: Q is a 5-basket. Let $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ be a 5-basket partition of the 5-basket Q, returned by the algorithm from Lemma 4.3. We now find indices $i^*, j^* \in \{1, 2, 3\}$ such that A is complete to $(B_1 \cup B_2 \cup B_3) \setminus B_{i^*}$, and F is complete to $V(Q) \setminus (B_{j^*} \cup C_{j^*})$ and anticomplete to $B_{j^*} \cup C_{j^*}$; such indices exist by the definition of a 5-basket, and they can be found in $O(n^2)$ time. Further, after possibly permuting indices, we may assume that either $i^* = j^* = 1$, or $i^* = 1$ and $j^* = 3$ (see Figure 3.3).

Next, we compute the degrees (in Q) of all vertices in A, and we order A as $A = \{a_1, \dots, a_t\}$ so that $d_Q(a_t) \leq \dots \leq d_Q(a_1)$; this can be done in $O(n^2)$ time. Since we already know that $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ is a 5-basket partition of the 5-basket Q, we see that $N_Q(a_t) \cap B_1 \subseteq \dots \subseteq N_Q(a_1) \cap B_1 = B_1$. Now, for all $i \in \{1, \dots, t\}$, we form the set $A_i = \{a_1, \dots, a_i\} \cup (N_Q(a_i) \cap B_1)$; clearly, the list $A_1, \dots, A_t$ can be computed in $O(n^2)$ time. By the definition of a 5-basket partition, and by our ordering of A, we have that $A_1, \dots, A_t$ are all cliques.

Our goal is to find all maximal cliques of Q. For this, we consider two cases: when $F = \emptyset$, and when $F \neq \emptyset$.

Case 2.1: $F = \emptyset$. In this case, it is easy to see that all maximal cliques of Q are on the following list of $t+6$ cliques:[32] $A_1, \dots, A_t, A \cup B_2, A \cup B_3, B_1 \cup C_1, B_2 \cup C_2, B_3 \cup C_3, C_1 \cup C_2 \cup C_3$. Since sets $A, B_1, B_2, B_3, C_1, C_2, C_3$ are nonempty and pairwise disjoint, and since $|A| = t$, we see that $t + 6 \leq |V(Q)|$; thus, the number of cliques on our list is at most $|V(Q)|$. We can now determine in $O(n^3)$ time which of those cliques are maximal in Q.

Case 2.2: $F \neq \emptyset$. Recall that either $j^* = 1$ or $j^* = 3$. If $j^* = 1$ (see Figure 3.3, top), then it is easy to see that all maximal cliques of Q are on the following list of $t + 7$ cliques:[33] $A_1, \dots, A_t, A \cup B_2 \cup F, A \cup B_3 \cup F, B_1 \cup C_1, B_2 \cup C_2 \cup F, B_3 \cup C_3 \cup F, C_1 \cup C_2 \cup C_3, C_2 \cup C_3 \cup F$. On the other hand, if $j^* = 3$ (see Figure 3.3, bottom), then all maximal cliques of Q are on the following list of $t + 7$ cliques:[34] $A_1 \cup F, \dots, A_t \cup F, A \cup B_2 \cup F, A \cup B_3, B_1 \cup C_1 \cup F, B_2 \cup C_2 \cup F, B_3 \cup C_3, C_1 \cup C_2 \cup C_3, C_1 \cup C_2 \cup F$. In either case (*i.e.* both when $j^* = 1$ and when $j^* = 3$), our list contains $t + 7$ cliques. Since sets $A, B_1, B_2, B_3, C_1, C_2, C_3, F$ are nonempty and pairwise disjoint, and since $|A| = t$, we see that $t + 7 \leq |V(Q)|$; thus, the number of cliques on our list is at most $|V(Q)|$. We can now determine in $O(n^3)$ time which of those cliques are maximal in Q.

In either case (*i.e.* both in Case 2.1 and in Case 2.2), we have found a complete list (say, $E_1, \dots, E_r$) of maximal cliques of Q, and this list contains at most $|V(Q)|$ cliques. By Claim 2, all maximal cliques of G are on the following list of cliques: $D_1, \dots, D_s, E_1 \cup U, \dots, E_r \cup U$. Clearly, the number of cliques on this list is at most $s + |V(Q)| \leq n$. We now check in $O(n^3)$ time which of the cliques $D_1, \dots, D_s, E_1 \cup U, \dots, E_r \cup U$ are maximal in G, we return those maximal cliques, and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. Furthermore, if the algorithm returns all maximal cliques of G, then the number of cliques returned is at most n. □

Corollary 5.2. *Every $(4K_1, C_4, C_6, C_7)$-free graph G has at most $|V(G)|$ maximal cliques.*

[31] However, not all C_i^j's need be maximal cliques of Q.

[32] However, not all cliques on this list need be maximal in Q.

[33] However, not all cliques on this list need be maximal in Q.

[34] Once again, not all cliques on this list need be maximal in Q.

Proof. This follows immediately from Theorem 5.1. $\square$

Lemma 5.3. *Let v be a simplicial vertex of a graph G. Then $N_G[v]$ is one of the cliques of some minimum clique cover of G.*

Proof. Set $k := \overline{\chi}(G)$, and let $\{C_1, \dots, C_k\}$ be a minimum clique cover of G. By symmetry, we may assume that $v \in C_1$. Since C_1 is a clique, we see that $C_1 \subseteq N_G[v]$. Since v is simplicial, $N_G[v]$ is a clique. But now $\{N_G[v], C_2, \dots, C_k\}$ is a minimum clique cover of G. $\square$

Theorem 5.4. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *Either a minimum clique cover of G, or the true statement that G is not $(4K_1, C_4, C_6, C_7)$-free;*
- ***Running time:*** *$O(n^3)$.*

Proof. We begin by finding a maximal sequence $v_1, \dots, v_s$ ($s \geq 0$) of vertices of G such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$; this can be done in $O(n^3)$ time by calling the algorithm from Lemma 3.3 with input G. For each $i \in \{1, \dots, s\}$, we form the set $D_i = N_G[v_i] \setminus \{v_1, \dots, v_{i-1}\}$. Clearly, sets $D_1, \dots, D_s$ can be computed in $O(n^2)$ time, and they are all cliques.

We now form a sequence $i_0, \dots, i_\ell$ of indices, and a sequence $S_0, \dots, S_\ell$ of sets, as follows. We set $i_0 := 0$ and $S_0 := \emptyset$. Then, having formed sequences $i_0, \dots, i_j$ and $S_0, \dots, S_j$, we proceed as follows. If $v_{i_j+1}, \dots, v_s \in S_j$,[35] then we set $\ell := j$, and we terminate our sequences. Otherwise, we extend our sequences by letting i_{j+1} be the smallest index $i \geq i_j + 1$ such that $v_i \notin S_j$, and setting $S_{j+1} := S_j \cup D_{i_{j+1}}$. Clearly, the sequences $\{i_j\}$ and $\{S_j\}$ can be formed in $O(n^2)$ time. To simplify notation, we set $S := S_\ell$. By construction, we have that $v_1, \dots, v_s \in S$.

Claim 1. If $S = V(G)$, then $\{D_{i_1}, \dots, D_{i_\ell}\}$ is a minimum clique cover of G. If $S \subsetneqq V(G)$, then for any set $\{E_1, \dots, E_r\}$ of cliques of G such that $r \leq \overline{\chi}(G \setminus S)$ and $V(G) \setminus S \subseteq E_1 \cup \dots \cup E_r$, we have that $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{E_1, \dots, E_r\}$ is a minimum clique cover of G.

Proof of Claim 1. This follows from Lemma 5.3, by an easy induction. ♦

If $S = V(G)$, then we return $\{D_{i_1}, \dots, D_{i_\ell}\}$, and we stop; by Claim 1, this is correct. From now on, we assume that $S \subsetneqq V(G)$. We then check if $V(G) \setminus S$ is a clique in $O(n^2)$ time. If $V(G) \setminus S$ is a clique, then we return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{V(G) \setminus S\}$, and we stop; by Claim 1, this is correct. From now on, we assume that S is not a clique. Then we check if $G \setminus S$ is cobipartite,[36] and if so, we find a partition $\{E_1, E_2\}$ of $V(G) \setminus S$ into two cliques; this can be done in $O(n^2)$ time via (for example) breadth-first-search in the graph $\overline{G} \setminus S$. If we found such a partition $\{E_1, E_2\}$ of $V(G) \setminus S$, then we return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{E_1, E_2\}$, and we stop; by Claim 1, this is correct. From now on, we assume that $G \setminus S$ is not cobipartite, and consequently, $\overline{\chi}(G \setminus S) \geq 3$.

We now form the graph $H := G \setminus \{v_1, \dots, v_s\}$,[37] as well as the set U of all universal vertices of H; this takes $O(n^2)$ time. By the maximality of $v_1, \dots, v_s$, we know that H is not complete, and consequently, $U \subsetneqq V(H)$. We now form the graph $Q := H \setminus U$ in further $O(n^2)$ time. Next, we determine whether Q is a 5-crown or a 5-basket (or neither) by calling the algorithms from Lemmas 4.1 and 4.3 with input Q. If Q is neither a 5-crown nor a 5-basket, then we return the answer that G is not $(4K_1, C_4, C_6, C_7)$-free, and we stop; by Theorem 3.12, this is correct. From

[35]Note that if $i_j = s$, then this vacuously holds.

[36]A graph is *cobiparite* if its vertex set can be partitioned into two cliques. So, a graph is cobipartite if its complement is bipartite.

[37]Clearly, $G \setminus S$ is an induced subgraph of H.

now on, we assume that Q is either a 5-crown or a 5-basket, and that we have also obtained a relevant partition, as specified in Lemmas 4.1 and 4.3.

Suppose first that we have determined that Q is a 5-crown. Let $(X_0, \dots, X_4)$ be a 5-crown partition of Q, returned by the algorithm from Lemma 4.1. By the definition of a 5-crown, there exists an index $i^* \in \mathbb{Z}_5$ such that X_{i^*-1} is complete to X_{i^*-2}, and X_{i^*+1} is complete to X_{i^*+2}; clearly, such an index i^* can be found in $O(n^2)$ time. Then $X_{i^*} \cup U, X_{i^*-1} \cup X_{i^*-2}, X_{i^*+1} \cup X_{i^*+2}$ are cliques of G, and clearly, $V(G) \setminus S \subseteq (X_{i^*} \cup U) \cup (X_{i^*-1} \cup X_{i^*-2}) \cup (X_{i^*+1} \cup X_{i^*+2})$. We now return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{X_{i^*} \cup U, X_{i^*-1} \cup X_{i^*-2}, X_{i^*+1} \cup X_{i^*+2}\}$, and we stop. Since $\overline{\chi}(G \setminus S) \geq 3$, Claim 1 guarantees that this is correct.

From now on, we assume that Q is a 5-basket. Let $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ be a 5-basket partition of Q, returned by the algorithm from Lemma 4.3. We now compute indices i^*, j^*, as in the definition of a 5-basket; clearly, this can be done in $O(n^2)$ time. After possibly permuting indices, we may assume that either $i^* = j^* = 1$, or $i^* = 1$ and $j^* = 3$. Note that this implies that F is complete to $A \cup B_2 \cup C_2$.

We form sets $U' := U \setminus S$, $A' := A \setminus S$, and $F' := F \setminus S$, and for each $i \in \{1, 2, 3\}$, we form sets $B'_i := B_i \setminus S$ and $C'_i := C_i \setminus S$; further, we check if A' is complete to B'_1. All this can be done in $O(n^2)$ time.

Suppose first that at least one of the sets $A', B'_2, C'_2, C'_3, B'_3$ is empty. Since $G[A \cup B_2 \cup C_2 \cup C_3 \cup B_3]$ is a 5-hyperhole, it is now easy to see that $A' \cup B'_2 \cup C'_2 \cup C'_3 \cup B'_3$ can be partitioned into two cliques, call them E_1 and E_2, which can be found in $O(n)$ time by routine checking.[38] If $j^* = 1$, then we return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{E_1 \cup F' \cup U', E_2, B'_1 \cup C'_1\}$, and we stop. On the other hand, if $j^* = 3$, then we return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{E_1, E_2, B'_1 \cup C'_1 \cup F' \cup U'\}$, and we stop. Since $\overline{\chi}(G \setminus S) \geq 3$, Claim 1 guarantees that this is correct.

From now on, we assume that sets $A', B'_2, C'_2, C'_3, B'_3$ are all nonempty. If $B'_1 = \emptyset$, then we return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{A' \cup B'_2 \cup F' \cup U', B'_3, C'_1 \cup C'_2 \cup C'_3\}$, and we stop; since $\overline{\chi}(G \setminus S) \geq 3$, Claim 1 guarantees that this is correct. From now on, we further assume that $B'_1 \neq \emptyset$.[39]

Suppose that A' is not complete to B'_1. Then $G \setminus S$ contains an induced $C_5 + K_1$ (the disjoint union of C_5 and K_1),[40] and clearly, $\overline{\chi}(C_5 + K_1) = 4$. Thus, $\overline{\chi}(G \setminus S) \geq 4$. We now return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{A' \cup B'_2 \cup F' \cup U', B'_1, B'_3, C'_1 \cup C'_2 \cup C'_3\}$, and we stop. By Claim 1, this is correct.

From now on, we assume that A' is complete to B'_1. If $C'_1 = \emptyset$, then we return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{A' \cup B'_1 \cup U', B'_2 \cup C'_2 \cup F', B'_3 \cup C'_3\}$, and we stop; since $\overline{\chi}(G \setminus S) \geq 3$, Claim 1 guarantees that this is correct. Suppose now that $C'_1 \neq \emptyset$. Then $G \setminus S$ contains an induced 5-pyramid (see Figure 3.2);[41] it is easy to see that $\overline{\chi}(\text{5-pyramid}) = 4$, and we deduce that $\overline{\chi}(G \setminus S) \geq 4$. We now return $\{D_{i_1}, \dots, D_{i_\ell}\} \cup \{A' \cup B'_2 \cup F' \cup U', B'_1, B'_3, C'_1 \cup C'_2 \cup C'_3\}$, and we stop. By Claim 1, this is correct.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. □

6 Coloring $(4K_1, C_4, C_6, C_7)$-free graphs

In this section, we give an $O(n^3)$ time coloring algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs (see Theorem 6.8). Our algorithm relies on the decomposition theorem for $(4K_1, C_4, C_6, C_7)$-

[38]Indeed, if $A' = \emptyset$, then we can take $E_1 := B'_2 \cup C'_2$ and $E_2 := B'_3 \cup C'_3$. If one of B'_2, B'_3 is empty, then we can set $E_1 := A' \cup B'_2 \cup B'_3$ and $E_2 := C'_2 \cup C'_3$. If $C'_2 = \emptyset$, then we can set $E_1 := A' \cup B'_2$ and $E_2 := B'_3 \cup C'_3$. If $C'_3 = \emptyset$, then we can set $E_1 := A' \cup B'_3$ and $E_2 := B'_2 \cup C'_2$.

[39]We now have that, of the sets $A', B'_1, B'_2, B'_3, C'_1, C'_2, C'_3, F'$, the only ones that may possibly be empty are C'_1 and F'.

[40]To see this, we select nonadjacent vertices $a \in A'$ and $b_1 \in B'_1$, we select any vertices $b_2 \in B'_2$, $c_2 \in C'_2$, $c_3 \in C'_3$, and $b_3 \in B'_3$, and we observe that these six vertices induce a $C_5 + K_1$ in G.

[41]To see this, select one vertex out of each of $A', B'_1, B'_2, B'_3, C'_1, C'_2, C'_3$; then those seven vertices induce a 5-pyramid.

free graphs from [23] (see Theorem 3.12 of the present paper), as well as on the formula for the chromatic number of a ring from [21] (see Theorem 3.8 of the present paper).

Recall that the "basic" classes from Theorem 3.12 are the 5-baskets and 5-crowns. Note that an induced subgraph of a 5-crown need not be a 5-crown, that is, 5-crowns do not form a hereditary class.[42] For the purposes of recursion, we will introduce a hereditary class (that of "5-pseudocrowns") that contains all 5-crowns (see subsection 6.1).

This section is organized as follows. First, in subsection 6.1, we define 5-pseudocrowns, and we give an $O(n^3)$ time coloring algorithm for them (see Lemma 6.5). Then, in subsection 6.2, we give an $O(n^3)$ time coloring algorithm for 5-baskets (see Lemma 6.7); we remark that our coloring algorithm for 5-baskets uses the coloring algorithm for 5-pseudocrowns as a subroutine. Finally, in subsection 6.3, we give an $O(n^3)$ time coloring algorithm for $(4K_1, C_4, C_6, C_7)$-free graphs (see Theorem 6.8).

6.1 Coloring 5-pseudocrowns

A *5-pseudocrown* is a graph Q whose vertex set can be partitioned into five (possibly empty) sets, say X_0, X_1, X_2, X_3, X_4 (with indices understood to be in $\mathbb{Z}_5$),[43] such that the following two conditions are satisfied:

- for all $i \in \mathbb{Z}_5$, X_i can be ordered as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $X_i \subseteq N_Q[u_i^{|X_i|}] \subseteq \dots \subseteq N_Q[u_i^1] \subseteq X_{i-1} \cup X_i \cup X_{i+1}$;[44]
- for some index $i^* \in \mathbb{Z}_5$, we have that X_{i^*-1} is complete to X_{i^*-2}, and X_{i^*+1} is complete to X_{i^*+2}.

Under these circumstances, we also say that $(X_0, X_1, X_2, X_3, X_4)$ is a *5-pseudocrown partition* of the 5-pseudocrown Q. We remark that the first bullet point above implies that sets $X_0, \dots, X_4$ are (possibly empty) cliques.

It is clear that the class of 5-pseudocrowns is hereditary. Indeed, if Q is a 5-pseudocrown with 5-pseudocrown partition $(X_0, \dots, X_4)$, then for every nonempty set $Y \subseteq V(Q)$, we have that $Q[Y]$ is a 5-pseudocrown with 5-pseudocrown partition $(X_0 \cap Y, \dots, X_4 \cap Y)$. It is also clear that every 5-crown is a 5-pseudocrown. Furthermore, we have the following lemma.

Lemma 6.1. *Let Q be a 5-pseudocrown with an associated 5-pseudocrown partition $(X_0, \dots, X_4)$. Then both the following hold:*

(a) either Q has a simplicial vertex, or Q is a 5-crown with 5-crown partition $(X_0, \dots, X_4)$;
(b) every hole in Q intersects each of $X_0, \dots, X_4$ in exactly one vertex, and in particular, every hole in Q is of length five.

Proof. For all $i \in \mathbb{Z}_5$, let $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ be an ordering of X_i such that $X_i \subseteq N_Q[u_i^{|X_i|}] \subseteq \dots \subseteq N_Q[u_i^1] \subseteq X_{i-1} \cup X_i \cup X_{i+1}$, as in the definition of a 5-pseudocrown.[45]

We first prove (a). Suppose first that at least one of $X_0, \dots, X_4$ is empty. Since our graphs are nonnull, sets $X_0, \dots, X_4$ cannot all be empty; so, we may assume by symmetry that $X_0 = \emptyset$ and $X_1 \neq \emptyset$. It then follows from our orderings of the sets X_i that $u_1^{|X_1|}$ is simplicial in Q. Suppose now that $X_0, \dots, X_4$ are all nonempty. If for all $i \in \mathbb{Z}_5$, we have that $N_Q[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, then Q is a 5-crown with 5-crown partition $(X_0, \dots, X_4)$, and we are done. We may now assume by symmetry that u_0^1 is not complete to X_1; then $u_1^{|X_1|}$ is anticomplete to X_0, and we see that $u_1^{|X_1|}$ is simplicial in Q. This proves (a).

[42] A class $\mathcal{G}$ of graphs is *hereditary* if for every graph $G \in \mathcal{G}$, the class $\mathcal{G}$ contains all graphs isomorphic to an induced subgraph of G. The reader may have noticed that 5-baskets do not form a hereditary class, either, but this is of no consequence for our coloring algorithm.

[43] Since our graphs are nonnull, at least one of the sets $X_0, \dots, X_4$ must be nonempty.

[44] It is possible that X_i is empty; in this case, our ordering of X_i is simply the null ordering.

[45] If $X_i = \emptyset$, then our ordering of X_i is simply the null ordering.

It remains to prove (b). Let $v_1, \dots, v_s$ ($s \geq 0$) be a maximal sequence of pairwise distinct vertices of Q such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $Q \setminus \{v_1, \dots, v_{i-1}\}$. If $V(Q) = \{v_1, \dots, v_s\}$, then $v_1, \dots, v_s$ is a simplicial elimination ordering of Q; in this case, Theorem 3.2 guarantees that Q is chordal, so that (b) is vacuously true. So, assume that $\{v_1, \dots, v_s\} \subsetneqq V(Q)$, and set $R := Q \setminus \{v_1, \dots, v_s\}$. Since no hole contains a simplicial vertex, we see that no hole in Q contains any one of $v_1, \dots, v_s$. Thus, every hole in Q is in fact a hole in R. Now, for all $i \in \mathbb{Z}_5$, set $Y_i = X_i \cap V(R)$; then R is a 5-pseudocrown with 5-pseudocrown partition $(Y_0, \dots, Y_4)$. Furthermore, by the maximality of $v_1, \dots, v_s$, the 5-pseudocrown R has no simplicial vertices, and so by (a) applied to R, we have that R is in fact a 5-crown with 5-crown partition $(Y_0, \dots, Y_4)$. But now (b) follows from Lemma 3.7(a)-(b). □

We now give an outline of the remainder of this subsection. First, Lemma 6.2 gives a formula for the chromatic number of a 5-pseudocrown; this formula easily follows from the one for the chromatic number of a ring (see Theorem 3.8). Then, using the formula from Lemma 6.2, we construct an $O(n^2)$ time algorithm that computes the chromatic number of a 5-pseudocrown, when an associated 5-pseudocrown partition is part of the input (see Lemma 6.3). Next, we construct an $O(n^3)$ time coloring algorithm that computes an optimal coloring of a 5-pseudocrown, when an associated 5-crown partition is part of the input (see Lemma 6.4). The algorithm from Lemma 6.4 makes $O(n)$ calls to the $O(n^2)$ time algorithm from Lemma 6.3, in order to repeatedly identify a suitable color class of an optimal coloring of our input 5-pseudocrown. Finally, we use the $O(n^3)$ time algorithms from Lemmas 3.3 and 6.4 to construct a "robust" coloring algorithm for 5-pseudocrowns (see Lemma 6.5). More precisely, for any input graph G, the algorithm from Lemma 6.5 either computes an optimal coloring of G, or correctly determines that G is not a 5-pseudocrown.[46]

We now turn to the technical details. As usual, the *order* of a graph is the number of vertices that is contains.

Lemma 6.2. *Let Q be a 5-pseudocrown, and let h be a nonnegative integer that satisfies the following two properties:*

- *no 5-hyperhole of Q is of order greater than h;*[47]
- *either Q contains a 5-hyperhole of order h, or $h \leq 2\omega(Q)$.*

Then $\chi(Q) = \max\{\omega(Q), \lceil h/2 \rceil\}$.

Proof. We first show that $\chi(Q) \geq \max\{\omega(Q), \lceil h/2 \rceil\}$. Clearly, $\chi(Q) \geq \omega(Q)$. Let us show that $\chi(Q) \geq \lceil h/2 \rceil$. If Q contains a 5-hyperhole H of order h, then we have that $\alpha(H) = 2$ and $\chi(Q) \geq \chi(H) \geq \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil = \lceil h/2 \rceil$. On the other hand, if $h \leq 2\omega(Q)$, then $\chi(Q) \geq \omega(Q) = \left\lceil \frac{2\omega(Q)}{2} \right\rceil \geq \lceil h/2 \rceil$. We have now shown that $\chi(Q) \geq \max\{\omega(Q), \lceil h/2 \rceil\}$.

It remains to show that $\chi(Q) \leq \max\{\omega(Q), \lceil h/2 \rceil\}$. Let $v_1, \dots, v_s$ ($s \geq 0$) be a maximal sequence of pairwise distinct vertices of Q such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in $Q \setminus \{v_1, \dots, v_{i-1}\}$. Suppose first that $V(Q) = \{v_1, \dots, v_s\}$. Then $v_1, \dots, v_s$ is a simplicial elimination ordering of Q, and it follows (by Theorem 3.2) that Q is chordal and therefore (by Theorem 3.1) perfect. So, $\chi(Q) = \omega(Q) \leq \max\{\omega(Q), \lceil h/2 \rceil\}$.

From now on, we assume that $\{v_1, \dots, v_s\} \subsetneqq V(Q)$. Set $R := Q \setminus \{v_1, \dots, v_s\}$. Clearly, $\chi(Q) = \max\{\chi(R), \omega(Q)\}$.[48] So, it suffices to show that $\chi(R) \leq \max\{\omega(R), \lceil h/2 \rceil\}$, for it will

[46] It is possible that the algorithm computes an optimal coloring of G, even though G is not a 5-pseudocrown.
[47] Note that if Q contains no 5-hyperholes, then this condition is vacuously satisfied.
[48] Obviously, $\chi(Q) \geq \max\{\chi(R), \omega(Q)\}$. For the reverse inequality, we observe that, if we take any optimal coloring of R, and then extend it to a proper coloring of Q by greedily assigning colors to vertices $v_s, \dots, v_1$ (in that order), we obtain a proper coloring of Q that uses at most $\max\{\chi(R), \omega(Q)\}$ colors.

then immediately follow that $\chi(Q) = \max\{\chi(R), \omega(Q)\} \leq \max\{\omega(Q), \lceil h/2 \rceil\}$, which is what we need. Since Q is a 5-pseudocrown, so is its induced subgraph R. Furthermore, by the maximality of $v_1, \dots, v_s$, we see that R has no simplicial vertices, and so by Lemma 6.1(a), R is in fact a 5-crown. In particular, R is a 5-ring, and so by Theorem 3.8, we have that $\chi(R) = \max\left(\{\omega(R)\} \cup \left\{\left\lceil \frac{|V(H)|}{2} \right\rceil \mid H \text{ is a 5-hyperhole in } R\right\}\right)$. By hypothesis, any 5-hyperhole in Q (and therefore, any 5-hyperhole in R) is of order at most h, and we now deduce that $\chi(R) \leq \max\{\omega(R), \lceil h/2 \rceil\}$. This completes the argument. $\square$

Lemma 6.3. *There exists an algorithm with the following specifications:*

- ***Input:*** *A 5-pseudocrown Q and a 5-pseudocrown partition $(X_0, \dots, X_4)$ of Q;*
- ***Output:*** *$\chi(Q)$;*
- ***Running time:*** *$O(n^2)$.*

Proof. We first compute the degrees of all vertices in Q, and then for each $i \in \mathbb{Z}_5$, we order X_i as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $d_Q(u_i^{|X_i|}) \leq \dots \leq d_Q(u_i^1)$; this can be done in $O(n^2)$ time. Since we already know that $(X_0, \dots, X_4)$ is a 5-pseudocrown partition of Q, we see that for all $i \in \mathbb{Z}_5$, we have that $X_i \subseteq N_Q[u_i^{|X_i|}] \subseteq \dots \subseteq N_Q[u_i^1] \subseteq X_{i-1} \cup X_i \cup X_{i+1}$. By the definition of a 5-pseudocrown, there exists an index $i^* \in \mathbb{Z}_5$ such that X_{i^*-1} is complete to X_{i^*-2}, and X_{i^*+1} is complete to X_{i^*+2}; clearly, such an index i^* can be found in $O(n^2)$ time. By symmetry, we may assume that $i^* = 0$. We now compute integers $\omega_0, \dots, \omega_4$ and r_0, as follows:

- for all $i \in \mathbb{Z}_5$, $\omega_i := \max\left(\{0\} \cup \left\{j + |N_Q(u_i^j) \cap X_{i+1}| \mid j \in \{1, \dots, |X_i|\}\right\}\right)$;[49]
- $r_0 := \max\left(\{0\} \cup \left\{j + |N_Q(u_0^j) \cap (X_4 \cup X_1)| \mid j \in \{1, \dots, |X_0|\}\right\}\right)$;[50]

We now return the number $\max\{\omega_0, \dots, \omega_4, \lceil \frac{r_0+\omega_2}{2} \rceil\}$, and we stop.

Clearly, integers $\omega_0, \dots, \omega_4$ and r_0 can be computed in $O(n^2)$ time, and so the overall running time of the algorithm is $O(n^2)$. It remains to prove correctness, that is, we must show that $\chi(Q) = \max\{\omega_0, \dots, \omega_4, \lceil \frac{r_0+\omega_2}{2} \rceil\}$. In view of Lemma 6.2, it suffices to prove the following:

(1) $\omega(Q) = \max\{\omega_0, \dots, \omega_4\}$;
(2) no 5-hyperhole of Q is of order greater than $r_0 + \omega_2$;
(3) either Q contains a 5-hyperhole of order $r_0 + \omega_2$, or $r_0 + \omega_2 \leq 2\omega(Q)$.

It is clear that $\omega(Q) = \max\{\omega(Q[X_i \cup X_{i+1}]) \mid i \in \mathbb{Z}_5, X_i \neq \emptyset\}$. Furthermore, in view of our orderings of the sets X_i, it is easy to see that for all $i \in \mathbb{Z}_5$, if $X_i \neq \emptyset$, then $\omega(Q[X_i \cup X_{i+1}]) = \omega_i$, and if $X_i = \emptyset$, then $\omega_i = 0$. Thus, (1) holds. Similarly, (2) readily follows from the definition of r_0 and ω_2.[51]

It remains to prove (3). It is obvious that $r_0 \leq 2\omega(Q)$;[52] furthermore, by (1), we have that $\omega_2 \leq \omega(Q)$. So, if $r_0 = 0$ or $\omega_2 = 0$, then $r_0 + \omega_2 \leq 2\omega(Q)$, and we are done. From now on, we assume that both r_0 and ω_2 are strictly positive. Then there exists an index $j_0 \in \{1, \dots, |X_0|\}$ such that $r_0 = j_0 + |N_Q(u_0^{j_0}) \cap (X_4 \cup X_1)|$, and an index $j_2 \in \{1, \dots, |X_2|\}$

[49] If $X_i = \emptyset$, then $\left\{j + |N_Q(u_i^j) \cap X_{i+1}| \mid j \in \{1, \dots, |X_i|\}\right\} = \emptyset$ and $\omega_i = 0$.

[50] If $X_0 = \emptyset$, then $\left\{j + |N_Q(u_0^j) \cap (X_4 \cup X_1)| \mid j \in \{1, \dots, |X_0|\}\right\} = \emptyset$ and $r_0 = 0$.

[51] Let us check this. Fix a 5-hyperhole H in Q. By Lemma 6.1(b), $V(H)$ intersects each of the sets $X_0, \dots, X_4$. For all $i \in \mathbb{Z}_5$, let j_i be the largest index in $\{1, \dots, |X_i|\}$ satisfying $u_i^{j_i} \in V(H)$. Then $|V(H)| \leq j_0 + \dots + j_4$. Furthermore, $u_0^{j_0}, u_1^{j_1}, u_2^{j_2}, u_3^{j_3}, u_4^{j_4}, u_0^{j_0}$ is a 5-hole in Q, and (in view of our orderings of the sets X_i) it follows that $j_4 \leq |N_Q(u_0^{j_0}) \cap X_4|$, $j_1 \leq |N_Q(u_0^{j_0}) \cap X_1|$, and $j_3 \leq |N_Q(u_2^{j_2}) \cap X_3|$. But now $j_0 + j_1 + j_4 \leq r_0$ and $j_2 + j_3 \leq \omega_2$, and so $|V(H)| \leq r_0 + \omega_2$.

[52] Indeed, in view of our orderings of the sets X_i, we see that for all $j \in \{1, \dots, |X_0|\}$, sets $\{u_0^1, \dots, u_0^j\} \cup (N_Q(u_0^j) \cap X_4)$ and $\{u_0^1, \dots, u_0^j\} \cup (N_Q(u_0^j) \cap X_1)$ are both cliques of Q, and their union is of size $j + |N_Q(u_0^j) \cap (X_4 \cup X_1)|$. So, $r_0 \leq 2\omega(Q)$.

such that $\omega_2 = j_2 + |N_Q(u_2^{j_2}) \cap X_3|$. Let $Y = \Big(N_Q(u_0^{j_0}) \cap X_4\Big) \cup \{u_0^1, \dots, u_0^{j_0}\} \cup \Big(N_Q(u_0^{j_0}) \cap X_1\Big) \cup \{u_2^1, \dots, u_2^{j_2}\} \cup \Big(N_Q(u_2^{j_2}) \cap X_3\Big)$. Clearly, $|Y| = r_0 + \omega_2$. Now, recall that $i^* = 0$, so that X_1 is complete to X_2, and X_3 is complete to X_4. We now see that, if sets $N_Q(u_0^{j_0}) \cap X_4$, $N_Q(u_0^{j_0}) \cap X_1$, and $N_Q(u_2^{j_2}) \cap X_3$ are all nonempty, then $Q[Y]$ is a 5-hyperhole in Q of order $|Y| = r_0 + \omega_2$. On the other hand, if at least one of these three sets is empty, then Y is the union of two cliques, and it follows that $r_0 + \omega_2 = |Y| \leq 2\omega(Q)$. This proves (3), and we are done. $\square$

Lemma 6.4. *There exists an algorithm with the following specifications:*

- ***Input:*** *A 5-pseudocrown Q and a 5-pseudocrown partition $(X_0, \dots, X_4)$ of Q;*
- ***Output:*** *An optimal coloring of Q;*
- ***Running time:*** $O(n^3)$.

Proof. We first compute the degrees of all vertices in Q, and then for each $i \in \mathbb{Z}_5$, we order X_i as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $d_Q(u_i^{|X_i|}) \leq \dots \leq d_Q(u_i^1)$; this can be done in $O(n^2)$ time. Since we already know that $(X_0, \dots, X_4)$ is a 5-pseudocrown partition of Q, we see that for all $i \in \mathbb{Z}_5$, we have that $X_i \subseteq N_Q[u_i^{|X_i|}] \subseteq \dots \subseteq N_Q[u_i^1] \subseteq X_{i-1} \cup X_i \cup X_{i+1}$.

Suppose first that for some $i \in \mathbb{Z}_5$, we have that $X_i = \emptyset$. Then the following is a simplicial elimination ordering of Q: $u_{i+1}^{|X_{i+1}|}, \dots, u_{i+1}^1, u_{i+2}^{|X_{i+2}|}, \dots, u_{i+2}^1, u_{i+3}^{|X_{i+3}|}, \dots, u_{i+3}^1, u_{i+4}^{|X_{i+4}|}, \dots, u_{i+4}^1$. We can obtain an optimal coloring of Q by coloring Q greedily using the reverse of this simplicial elimination ordering; this takes $O(n^2)$ time. We return this coloring of Q, and we stop.

We may now assume that $X_0, \dots, X_4$ are all nonempty. We check in $O(1)$ time whether there exists an index $i \in \mathbb{Z}_5$ such that $u_i^{|X_i|}$ is nonadjacent to at least one of u_{i-1}^1, u_{i+1}^1. Suppose first that we have found such an index i. Then $u_i^{|X_i|}$ is a simplicial vertex.[53] We now recursively obtain an optimal coloring of $Q \setminus u_i^{|X_i|}$. Then, we extend this coloring to an optimal coloring of Q by assigning to $u_i^{|X_i|}$ a color used on $Q \setminus u_i^{|X_i|}$, but not on $N_Q(u_i^{|X_i|})$, if such a color exists, and otherwise assigning to $u_i^{|X_i|}$ a color not used on $Q \setminus u_i^{|X_i|}$; this takes $O(n)$ time. We return this coloring of Q, and we stop.

From now on, we assume that for all $i \in \mathbb{Z}_5$, the vertex $u_i^{|X_i|}$ is adjacent to both u_{i-1}^1 and u_{i+1}^1. In view of our orderings of the sets X_i, this implies that for all $i \in \mathbb{Z}_5$, u_i^1 is complete to $X_{i-1} \cup X_{i+1}$. By the definition of a 5-pseudocrown, there exists some $i \in \mathbb{Z}_5$ such that X_i is complete to X_{i+1}, and clearly, such an i can be found in $O(n^2)$ time; by symmetry, we may assume that X_2 is complete to X_3 (and so $X_2 \cup X_3$ is a clique). We note that $N_Q[u_0^1] = X_4 \cup X_0 \cup X_1$, and so the nonneighborhood of u_0^1 in Q is $V(Q) \setminus N_Q[u_0^1] = X_2 \cup X_3$. Now, we compute $\chi_2 := \chi(Q \setminus \{u_0^1, u_2^1\})$ and $\chi_3 := \chi(Q \setminus \{u_0^1, u_3^1\})$ by calling the $O(n^2)$ time algorithm from Lemma 6.3.[54]

Claim 1. $\chi(Q) = \min\{\chi_2, \chi_3\} + 1$.

Proof of Claim 1. By symmetry, we may assume that $\chi_2 \leq \chi_3$. It is obvious that $\chi(Q) \leq \chi_2 + 1$: indeed, any optimal coloring of $Q \setminus \{u_0^1, u_2^1\}$ can be extended to a proper coloring of Q that uses only $\chi_2 + 1$ colors simply by assigning the same new color (*i.e.* a color not used on $Q \setminus \{u_0^1, u_2^1\}$) to the nonadjacent vertices u_0^1 and u_2^1. It remains to show that $\chi(Q) \geq \chi_2 + 1$. Consider any optimal coloring of Q, and let S be the color class of this coloring that contains the vertex u_0^1. Clearly, $\chi(Q \setminus S) = \chi(Q) - 1$, and so we need only show that $\chi(Q \setminus S) \geq \chi_2$. Since

[53]Indeed, if $u_i^{|X_i|}$ is nonadjacent to u_{i-1}^1, then it follows from our orderings of the sets $X_0, \dots, X_4$ that $u_i^{|X_i|}$ is anticomplete to X_{i-1}, and that $u_i^{|X_i|}$ is simplicial in Q. The argument is analogous in the case when $u_i^{|X_i|}$ is nonadjacent to u_{i+1}^1.

[54]For χ_2, the input is $Q \setminus \{u_0^1, u_2^1\}$ and $(X_0 \setminus \{u_0^1\}, X_1, X_2 \setminus \{u_2^1\}, X_3, X_4)$. For χ_3, the input is $Q \setminus \{u_0^1, u_3^1\}$ and $(X_0 \setminus \{u_0^1\}, X_1, X_2, X_3 \setminus \{u_3^1\}, X_4)$.

$V(Q) \setminus N_Q[u_0^1] = X_2 \cup X_3$, we see that $S \subseteq \{u_0^1\} \cup X_2 \cup X_3$. If $S \cap (X_2 \cup X_3) = \emptyset$, then $S = \{u_0^1\}$, and so $\chi(Q \setminus S) \geq \chi(Q \setminus \{u_0^1, u_2^1\}) = \chi_2$, which is what we needed. Suppose now that $S \cap (X_2 \cup X_3) \neq \emptyset$. Since S is a stable set and $X_2 \cup X_3$ is a clique, we have that $|S \cap (X_2 \cup X_3)| = 1$. Fix $i \in \{2, 3\}$ and $j \in \{1, \dots, |X_i|\}$ such that $S \cap (X_2 \cup X_3) = \{u_i^j\}$; then $S = \{u_0^1, u_i^j\}$. It then follows from our ordering of X_i that $Q \setminus \{u_0^1, u_i^1\}$ is isomorphic to a (not necessarily induced) subgraph of $Q \setminus S$; consequently, $\chi(Q \setminus S) \geq \chi(Q \setminus \{u_0^1, u_i^1\}) = \chi_i \geq \chi_2$. This proves Claim 1. ♦

We now complete the description of the algorithm. We compare the numbers χ_2 and χ_3; by symmetry, we may assume that $\chi_2 \leq \chi_3$, so that by Claim 1, we have that $\chi(Q) = \chi_2 + 1$. We now recursively obtain an optimal coloring of $Q \setminus \{u_0^1, u_2^1\}$ (this coloring uses precisely χ_2 colors), and we extend it to an optimal coloring of Q by assigning the same new color (*i.e.* a color not used on $Q \setminus \{u_0^1, u_2^1\}$) to the nonadjacent vertices u_0^1 and u_2^1. We return this coloring of Q, and we stop.

Clearly, the algorithm is correct. We make $O(n)$ recursive calls to the algorithm, and otherwise, the slowest step takes $O(n^2)$ time. So, the total running time of the algorithm is $O(n^3)$. □

Lemma 6.5. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *Either an optimal coloring of G, or the true statement that G is not a 5-pseudocrown;*
- ***Running time:*** $O(n^3)$.

Proof. We first form a maximal sequence $v_1, \dots, v_s$ ($s \geq 0$) of vertices of G such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$; this can be done in $O(n^3)$ time by calling the algorithm from Lemma 3.3 with input G. If $V(G) = \{v_1, \dots, v_s\}$, then $v_1, \dots, v_s$ is a simplicial elimination ordering of G; in this case, we obtain an optimal coloring of G in $O(n^2)$ time by coloring its vertices greedily using the ordering $v_s, \dots, v_1$, we return this coloring, and we stop.

We may now assume that $\{v_1, \dots, v_s\} \subsetneqq V(G)$. We then form the graph $Q := G \setminus \{v_1, \dots, v_s\}$ in $O(n^2)$ time. By the maximality of $v_1, \dots, v_s$, the graph Q has no simplicial vertices. We call the $O(n^2)$ time algorithm from Lemma 4.1 with input Q. If the algorithm returned the answer that Q is not a 5-crown, then Lemma 6.1(a) guarantees that Q is not a 5-pseudocrown; in this case, G is not a 5-pseudocrown either, and we return this answer and stop. Suppose now that the algorithm returned the answer that Q is a 5-crown, together with an associated 5-crown partition $(X_0, \dots, X_4)$. We then obtain an optimal coloring of Q by calling the $O(n^3)$ time algorithm from Lemma 6.4 with input Q and $(X_0, \dots, X_4)$, and we extend this coloring to an optimal coloring of G by greedily assigning colors to vertices $v_s, \dots, v_1$ (in that order) in further $O(n^2)$ time. We return this coloring of G, and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. □

6.2 Coloring 5-baskets

In this section, we give an $O(n^3)$ time coloring algorithm for 5-baskets (see Lemma 6.7). We begin with a technical lemma (Lemma 6.6) on which this coloring algorithm relies. Before going into technical details, let us give the idea behind Lemma 6.6. Obviously, for any graph Q, and any nonempty set $X \subsetneqq V(Q)$,[55] we have that $\chi(Q) \leq \chi(Q[X]) + \chi(Q \setminus X)$. Roughly speaking, for a 5-basket Q, Lemma 6.6 identifies $O(n^2)$ sets X, for at least one of which the trivial inequality $\chi(Q) \leq \chi(Q[X]) + \chi(Q \setminus X)$ becomes an equality. Let us be a bit more

[55] Recall that all our graphs are assumed to be nonnull. So, for $Q[X]$ and $Q \setminus X$ to both be defined, we need that $\emptyset \neq X \subsetneqq V(Q)$.

precise. Suppose that Q is a 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$. Suppose furthermore that either $i^* = j^* = 1$, or $i^* = 1$ and $j^* = 3$, where i^*, j^* are as in the definition of a 5-basket (see Figure 3.3). Lemma 6.6 specifies a family $\{X_{j,k}\}$ of $O(n^2)$ proper, nonempty subsets of $V(Q)$, each of which induces a chordal subgraph of Q of chromatic number $|B_2|$.[56] Furthermore, for at least one ("optimal") choice of indices j, k, we have that $\chi(Q) = \chi(Q[X_{j,k}]) + \chi(Q \setminus X_{j,k}) = |B_2| + \chi(Q \setminus X_{j,k})$. Our coloring algorithm for 5-baskets (see Lemma 6.7) uses Lemma 6.6 as follows. First, the algorithm identifies the optimal pair j, k. Then, for that optimal pair, it computes optimal colorings of the graphs $Q[X_{j,k}]$ and $Q \setminus X_{j,k}$ (these two colorings use disjoint color sets). Finally, an optimal coloring of Q is obtained by taking the union of these two colorings.

Lemma 6.6. *Let Q be a 5-basket with an associated 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$. Assume that either $i^* = j^* = 1$, or $i^* = 1$ and $j^* = 3$, where i^*, j^* are as in the definition of a 5-basket. Let $B_1 = \{b_1, \dots, b_p\}$ be an ordering of B_1 such that $N_Q[b_p] \subseteq \dots \subseteq N_Q[b_1]$.[57] Further, for all indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$, set $\gamma_{j,k} := \min\{|C_1|, |B_2| - j, |B_2| - k\}$ and $\beta_k := \min\{|B_3|, |B_2| - k\}$. For indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$, a set $X_{j,k} \subseteq V(Q)$ is said to be* (j,k)-good *if it satisfies the following two properties (see Figure 6.1):*

- *$B_2 \cup \{b_1, \dots, b_j\} \subseteq X_{j,k} \subseteq B_2 \cup \{b_1, \dots, b_j\} \cup C_1 \cup C_3 \cup B_3$;*
- *$|C_1 \cap X_{j,k}| = \gamma_{j,k}$, $|C_3 \cap X_{j,k}| = k$, and $|B_3 \cap X_{j,k}| = \beta_k$.*

Then both the following hold:

(a) for all indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$, and all (j,k)-good sets $X_{j,k}$, the graph $Q[X_{j,k}]$ is chordal and satisfies $\chi(Q[X_{j,k}]) = \omega(Q[X_{j,k}]) = |B_2|$;

(b) there exist indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$ such that all (j,k)-good sets $X_{j,k}$ satisfy $\chi(Q) = |B_2| + \chi(Q \setminus X_{j,k})$.

Proof. We first prove (a). Fix indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$, and let $X_{j,k} \subseteq V(Q)$ be any (j,k)-good set. We can obtain a simplicial elimination ordering of $Q[X_{j,k}]$ by first listing all vertices of $(B_1 \cup B_2 \cup B_3) \cap X_{j,k}$ (in any order), and then listing all vertices of $(C_1 \cup C_3) \cap X_{j,k}$ (in any order). So, by Theorem 3.2, $Q[X_{j,k}]$ is chordal. Theorem 3.1 now implies that $Q[X_{j,k}]$ is perfect, and consequently, $\chi(Q[X_{j,k}]) = \omega(Q[X_{j,k}])$. The fact that $\omega(Q[X_{j,k}]) = |B_2|$ is immediate from the construction. This proves (a).

It remains to prove (b). To simplify notation, we set $\chi := \chi(Q)$. Let $\{S_1, \dots, S_\chi\}$ be a partition of $V(Q)$ into stable sets.[58] Since B_2 is a clique, we see that exactly $|B_2|$ of the sets $S_1, \dots, S_\chi$ intersect B_2; by symmetry, we may assume that $S_1, \dots, S_{|B_2|}$ all intersect B_2,[59] and that $S_{|B_2|+1}, \dots, S_\chi$ do not intersect B_2. Set $X := S_1 \cup \dots \cup S_{|B_2|}$. Since B_2 is complete to $A \cup C_2 \cup F$ and anticomplete to $B_1 \cup C_1 \cup C_3 \cup B_3$, we have that $B_2 \subseteq X \subseteq B_2 \cup B_1 \cup C_1 \cup C_3 \cup B_3$. For each $i \in \{1, 3\}$, we define sets $B_i' := B_i \cap X$ and $C_i' := C_i \cap X$. Then $X = B_2 \cup B_1' \cup C_1' \cup C_3' \cup B_3'$. Furthermore, $\{S_1, \dots, S_{|B_2|}\}$ is a partition of X into stable sets of Q, and $\{S_{|B_2|+1}, \dots, S_\chi\}$ is a partition of $V(Q) \setminus X$ into stable sets of Q. Since $S_1, \dots, S_\chi$ are the color classes of an optimal coloring of Q, we see that $\chi(Q[X]) = |B_2|$ and $\chi(Q \setminus X) = \chi - |B_2|$. Moreover, it is clear that $\omega(G[X]) = |B_2|$.[60] Now, let $j := |B_1'|$ and $k := |C_3'|$. Since $B_1' \cup C_1', C_1' \cup C_3', C_3' \cup B_3'$ are all

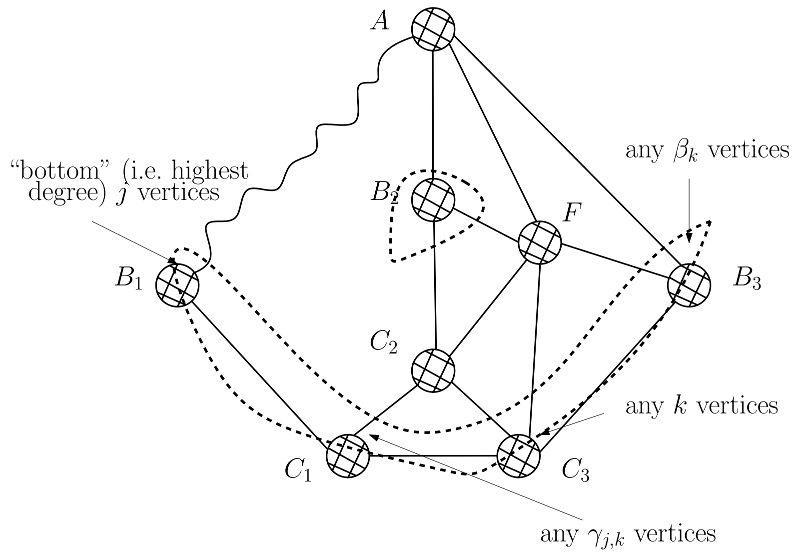
[56]We also note that $Q \setminus X_{j,k}$ is in fact a 5-pseudocrown, though Lemma 6.6 does not address this. See, instead, Claim 1 from the proof of Lemma 6.7. The important point is that $Q \setminus X_{j,k}$ can be colored using the algorithm from Lemma 6.5.

[57]Note that such an ordering exists by the definition of a 5-basket.

[58]So, $S_1, \dots, S_\chi$ are the color classes of some optimal coloring of Q.

[59]Note that $|S_i \cap B_2| = 1$ for all $i \in \{1, \dots, |B_2|\}$.

[60]Indeed, B_2 is a clique of $G[X]$, and so $\omega(G[X]) \geq |B_2|$. On the other hand, $\omega(G[X]) \leq \chi(G[X]) = |B_2|$.



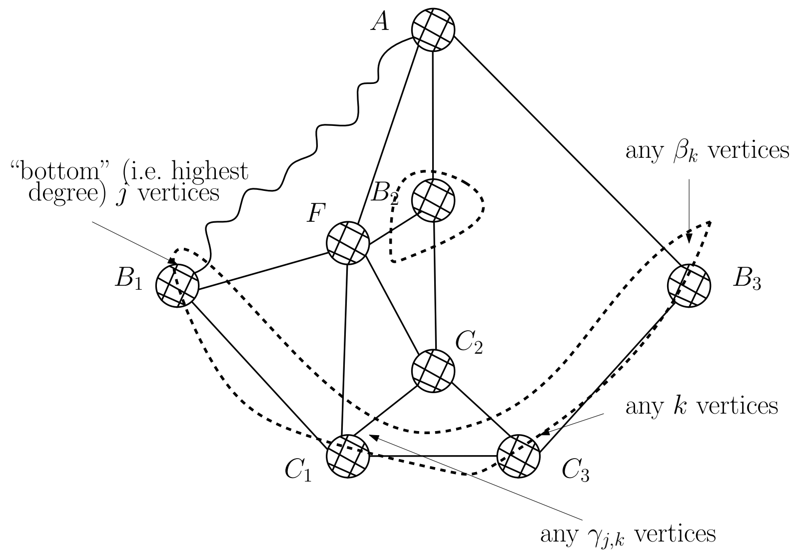


Figure 6.1: A 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$, and with $i^* = j^* = 1$ (top), or $i^* = 1$ and $j^* = 3$ (bottom). In both cases, a (j, k)-good set $X_{j,k}$ is the union of sets represented by the two dashed bags.

cliques of $G[X]$, and are therefore of size at most $\omega(G[X]) = |B_2|$, we see that all the following hold: $0 \leq j \leq \min\{|B_1|, |B_2|\}$, $0 \leq k \leq \min\{|C_3|, |B_2|\}$, $|C_1'| \leq \gamma_{j,k}$, and $|B_3'| \leq \beta_k$.

Now, fix any (j,k)-good set $X_{j,k}$. By (a), $Q[X_{j,k}]$ is a chordal graph with $\chi(Q[X_{j,k}]) = |B_2|$, and it follows that $\chi \leq \chi(Q[X_{j,k}]) + \chi(Q \setminus X_{j,k}) = |B_2| + \chi(Q \setminus X_{j,k})$. On the other hand, we note that $Q \setminus X_{j,k}$ is isomorphic to a (not necessarily induced) subgraph of $Q \setminus X$, and consequently, $\chi(Q \setminus X_{j,k}) \leq \chi(Q \setminus X) = \chi - |B_2|$; thus, $\chi \geq |B_2| + \chi(Q \setminus X_{j,k})$. It now follows that $\chi = |B_2| + \chi(Q \setminus X_{j,k})$, and we are done. □

Lemma 6.7 (below) gives an $O(n^3)$ time coloring algorithm for 5-baskets. Before going into technical details, let us give a brief outline of the algorithm. We use the notation from Lemma 6.6. For all indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$, the graph $Q[X_{j,k}]$ is chordal with chromatic number $|B_2|$, and $Q \setminus X_{j,k}$ is a 5-pseudocrown (see Claim 1 in the proof of Lemma 6.7). We compute numbers $r_{j,k}$ and $h_{j,k}$ such that $\omega(Q \setminus X_{j,k}) = r_{j,k}$ and $\chi(Q \setminus X_{j,k}) = \max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\}$, as in Lemma 6.2. Crucially, it is possible to compute the families $\{r_{j,k}\}$ and $\{h_{j,k}\}$ in $O(n^3)$ time, without actually computing the family of sets $\{X_{j,k}\}$ and the related induced subgraphs of Q. We then find the indices j,k for which $\max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\}$ is minimum, and only for that "optimal" pair of indices j,k do we compute the set $X_{j,k}$ and the induced subgraphs $Q[X_{j,k}]$ and $Q \setminus X_{j,k}$ of Q. We color the 5-pseudocrown $Q \setminus X_{j,k}$ using the $O(n^3)$ time algorithm from Lemma 6.5, and we extend that to a proper coloring of Q by coloring the chordal graph $Q[X_{j,k}]$ with $|B_2|$ new colors. Lemma 6.6 guarantees that this coloring of Q is optimal.

Lemma 6.7. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph Q;*
- ***Output:*** *Either an optimal coloring of Q, or the true statement that Q is not a 5-basket;*
- ***Running time:*** *$O(n^3)$.*

Proof. We first call the $O(n^2)$ time algorithm from Lemma 4.3 with input Q. If the algorithm returns the answer that Q is not a 5-basket, then we return this answer as well, and we stop. From now on, we assume that the algorithm returned the answer that Q is a 5-basket, together with an associated 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$. We then find indices $i^*, j^* \in \{1,2,3\}$ such that A is complete to $(B_1 \cup B_2 \cup B_3) \setminus B_{i^*}$, and F is complete to $V(Q) \setminus (B_{j^*} \cup C_{j^*} \cup F)$ and anticomplete to $B_{j^*} \cup C_{j^*}$, as in the definition of a 5-basket; this can be done in $O(n^2)$ time. After possibly permuting indices, we may assume that either $i^* = j^* = 1$, or $i^* = 1$ and $j^* = 3$ (see Figure 3.3).

Now, we compute the degrees of all vertices in $A \cup B_1$, we order A as $A = \{a_1, \dots, a_t\}$ so that $d_Q(a_t) \leq \dots \leq d_Q(a_1)$, and we order B_1 as $B_1 = \{b_1, \dots, b_p\}$ so that $d_Q(b_p) \leq \dots \leq d_Q(b_1)$; this takes $O(n^2)$ time. Since we already know that $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ is a 5-basket partition of Q, we see that $N_G(a_t) \cap B_1 \subseteq \dots \subseteq N_G(a_1) \cap B_1 = B_1$ and $a_1 \in N_Q(b_p) \cap A \subseteq \dots \subseteq N_Q(b_1) \cap A$.

For all $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $\ell \in \{1, \dots, t\}$, let $q_j(a_\ell) = |N_Q(a_\ell) \cap \{b_{j+1}, \dots, b_p\}|$. Further, for all $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$, let $\ell_j \in \{1, \dots, t\}$ be such that for all $\ell \in \{1, \dots, t\}$, we have that $\ell_j + q_j(a_{\ell_j}) \geq \ell + q_j(a_\ell)$; to simply notation, for each $j \in \{1, \dots, t\}$, we set $p_j := q_j(a_{\ell_j})$. It follows immediately from the orderings of A and B_1 that, for all indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$, the set $\{a_1, \dots, a_{\ell_j}\} \cup \{b_{j+1}, \dots, b_{j+p_j}\}$ is a clique of maximum size in $Q[A \cup \{b_{j+1}, \dots, b_p\}]$,[61] and consequently,

$$\omega(Q[A \cup \{b_{j+1}, \dots, b_p\}]) \quad = \quad \ell_j + p_j.$$

[61]Here, we are also using the fact that a_1 is a universal vertex of $Q[A \cup B_1]$, and consequently, any maximal clique of $Q[A \cup \{b_{j+1}, \dots, b_p\}]$ contains a_1.

Next, for all $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$, we set $\gamma_{j,k} := \min\{|C_1|, |B_2| - j, |B_2| - k\}$ and $\beta_k := \min\{|B_3|, |B_2| - k\}$, as in the statement of Lemma 6.6. Clearly, families $\{q_j\}$, $\{\ell_j\}$, $\{p_j\}$, $\{\gamma_{j,k}\}$, and $\{\beta_k\}$ can all be computed in $O(n^3)$ time.

Recall that $t = |A|$. Now, if $j^* = 1$ (see Figure 6.1, top), then for all $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|B_2|, |C_3|\}\}$, we set:

- $r_0^{j,k} := \Big(|B_1| - j\Big) + \Big(|C_1| - \gamma_{j,k}\Big)$;
- $r_1^{j,k} := \Big(|C_1| - \gamma_{j,k}\Big) + |C_2| + \Big(|C_3| - k\Big)$;
- $r_2^{j,k} := \Big(|C_3| - k\Big) + |F| + \max\Big\{|C_2|, |B_3| - \beta_k\Big\}$;
- $r_3^{j,k} := t + \Big(|B_3| - \beta_k\Big) + |F|$;
- $r_4^{j,k} := \ell_j + p_j$;
- $r_{j,k} := \max\{r_0^{j,k}, r_1^{j,k}, r_2^{j,k}, r_3^{j,k}, r_4^{j,k}\}$;
- $h_{j,k} := \Big(\ell_j + p_j\Big) + \Big(|C_1| - \gamma_{j,k}\Big) + \Big(|C_3| - k\Big) + |F| + \max\Big\{|C_2|, |B_3| - \beta_k\Big\}$.

On the other hand, if $j^* = 3$ (see Figure 6.1, bottom), then for all $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|B_2|, |C_3|\}\}$, we set:

- $r_0^{j,k} := \max\{|B_1| - j, |C_2|\} + \Big(|C_1| - \gamma_{j,k}\Big) + |F|$;
- $r_1^{j,k} := \Big(|C_1| - \gamma_{j,k}\Big) + |C_2| + \Big(|C_3| - k\Big)$;
- $r_2^{j,k} := \Big(|C_3| - k\Big) + \Big(|B_3| - \beta_k\Big)$;
- $r_3^{j,k} := t + \Big(|B_3| - \beta_k\Big)$;
- $r_4^{j,k} := \ell_j + p_j + |F|$;
- $r_{j,k} := \max\{r_0^{j,k}, r_1^{j,k}, r_2^{j,k}, r_3^{j,k}, r_4^{j,k}\}$;
- $h_{j,k} := \max\Big\{\ell_j + p_j, t + |C_2|\Big\} + |F| + \Big(|C_1| - \gamma_{j,k}\Big) + \Big(|C_3| - k\Big) + \Big(|B_3| - \beta_k\Big)$.

Clearly, families $\{r_i^{j,k}\}$, $\{r_{j,k}\}$, and $\{h_{j,k}\}$ can be computed in $O(n^3)$ time.

Before completing the description of the algorithm, we state and prove a claim that we will use in the proof of correctness of our algorithm. A "(j,k)-good set" is defined as in Lemma 6.6 (see Figure 6.1).

> **Claim 1.** For all indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$, and all (j,k)-good sets $X_{j,k}$, the graph $Q_{j,k} := Q \setminus X_{j,k}$ is a 5-pseudocrown, and it satisfies $\omega(Q \setminus X_{j,k}) = r_{j,k}$ and $\chi(Q \setminus X_{j,k}) = \max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\}$.

Proof of Claim 1. We fix indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$, and we let $X_{j,k}$ be any (j,k)-good set. We set $Q_{j,k} := Q \setminus X_{j,k}$, as in the statement of the claim.

Now, we define sets $Y_0, \dots, Y_4$, with indices in $\mathbb{Z}_5$, as follows (see Figure 6.2). If $j^* = 1$, then we set $Y_0 := B_1$, $Y_1 := C_1$, $Y_2 := C_2 \cup C_3$, $Y_3 := B_3 \cup F$, $Y_4 := A$. On the other hand, if $j^* = 3$, then we set $Y_0 := B_1 \cup F$, $Y_1 := C_1 \cup C_2$, $Y_2 := C_3$, $Y_3 := B_3$, $Y_4 := A$.

Note that $Q \setminus B_2$ is a 5-pseudocrown with 5-pseudocrown partition $(Y_0, \dots, Y_4)$, and furthermore, Y_1 is complete to Y_2, and Y_3 is complete to Y_4. Now, for all $i \in \mathbb{Z}_5$, we set $Z_i := Y_i \setminus X_{j,k}$. Then $Q_{j,k}$ is a 5-pseudocrown with 5-pseudocrown partition $(Z_0, \dots, Z_4)$, and furthermore, Z_1 is complete to Z_2, and Z_3 is complete to Z_4.

Clearly, $\omega(Q_{j,k}) = \max\{\omega(Q_{j,k}[Z_i \cup Z_{i+1}]) \mid i \in \mathbb{Z}_5, Z_i \cup Z_{i+1} \neq \emptyset\}$. Further, it is clear from the construction that for all $i \in \mathbb{Z}_5$ such that $Z_i \cup Z_{i+1} \neq \emptyset$, we have that $\omega(Q_{j,k}[Z_i \cup Z_{i+1}]) = r_i^{j,k}$.[62] Thus, $\omega(Q_{j,k}) = r_{j,k}$.

[62] If $Z_i \cup Z_{i+1} = \emptyset$, then $r_i^{j,k} = 0$.

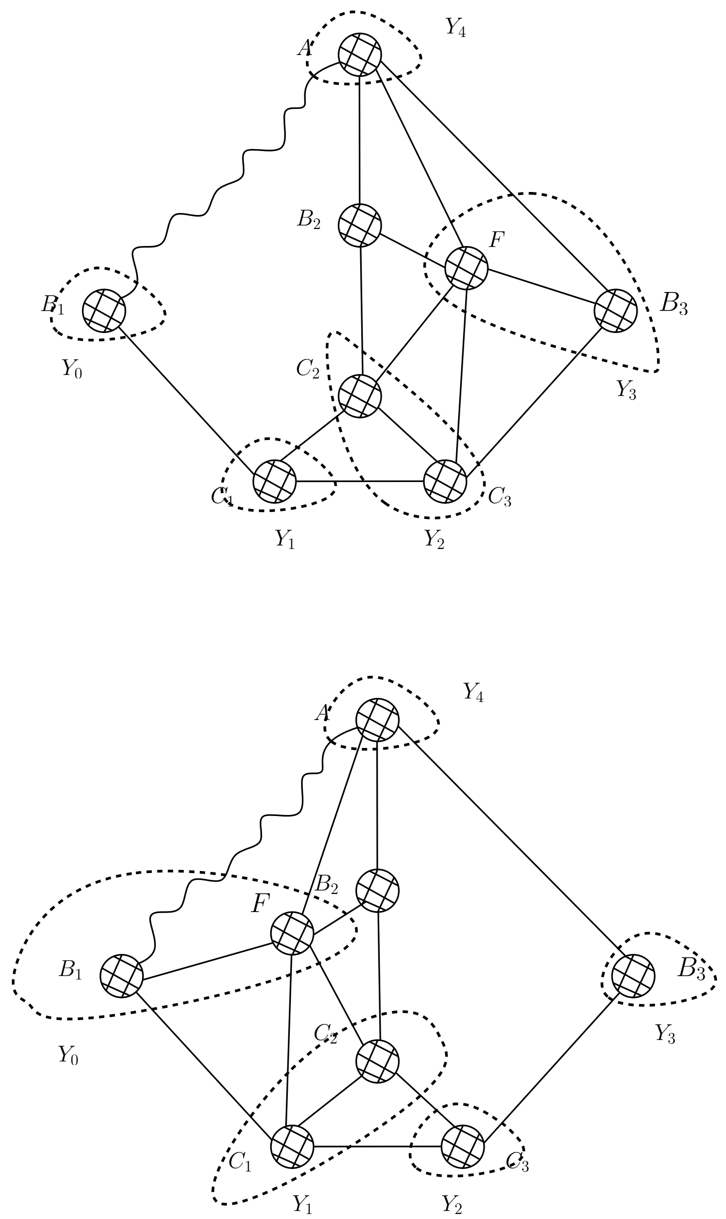


Figure 6.2: A 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$, and with $i^* = j^* = 1$ (top), or $i^* = 1$ and $j^* = 3$ (bottom). $Q \setminus B_2$ is a 5-pseudocrown with 5-pseudocrown partition $(Y_0, \ldots, Y_4)$, with Y_1 complete to Y_2, and Y_3 complete to Y_4; sets $Y_0, \ldots, Y_4$ are represented by dashed bags.

It remains to show that $\chi(Q_{j,k}) = \max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\}$. In view of Lemma 6.2, it suffices to show that $h_{j,k}$ satisfies the following:

(a) no 5-hyperhole of $Q_{j,k}$ is of order greater than $h_{j,k}$;
(b) either $Q_{j,k}$ contains a 5-hyperhole of order $h_{j,k}$, or $h_{j,k} \leq 2\omega(Q_{j,k})$.

We first prove (a). Suppose that H is a 5-hyperhole in Q. Since Q is a 5-pseudocrown with 5-pseudocrown partition $(Z_0, \dots, Z_4)$, Lemma 6.1(b) guarantees that $V(H)$ intersects each of $Z_0, \dots, Z_4$, and furthermore, that for all $i \in \mathbb{Z}_5$, $V(H) \cap Z_i$ is complete to $V(H) \cap Z_{i+1}$. We consider two cases: when $j^* = 1$, and when $j^* = 3$.

Suppose first that $j^* = 1$. Since $V(H) \cap Z_4$ is complete to $V(H) \cap Z_0$, we see that $|V(H) \cap (Z_0 \cup Z_4)| \leq \ell_j + p_j$. Further, since $V(H) \cap Z_2$ is complete to $V(H) \cap Z_3$, we see that either $V(H) \cap C_2 = \emptyset$ or $V(H) \cap B_3 = \emptyset$, and we deduce that $|V(H) \cap (Z_1 \cup Z_2 \cup Z_3)| \leq |C_1 \setminus X_{j,k}| + |C_3 \setminus X_{j,k}| + |F| + \max\{|C_2|, |B_3 \setminus X_{j,k}|\}$. It now readily follows that $|V(H)| \leq h_{j,k}$.

Suppose now that $j^* = 3$. Suppose first that $V(H) \cap \{b_{j+1}, \dots, b_p\} \neq \emptyset$. Since $V(H) \cap Z_0$ is complete to $V(H) \cap Z_4$, we deduce that $|V(H) \cap (Z_0 \cup Z_4)| \leq \ell_j + p_j + |F|$. Further, since $V(H) \cap Z_0$ is complete to $V(H) \cap Z_1$, the fact that $V(H) \cap \{b_{j+1}, \dots, b_p\} \neq \emptyset$ implies that $V(H) \cap C_2 = \emptyset$; consequently, $V(H) \cap (Z_1 \cup Z_2 \cup Z_3) \subseteq (C_1 \setminus X_{j,k}) \cup (C_3 \setminus X_{j,k}) \cup (B_3 \setminus X_{j,k})$, and it readily follows that $|V(H)| \leq h_{j,k}$. Suppose now that $V(H) \cap \{b_{j+1}, \dots, b_p\} = \emptyset$. Then $V(H) \subseteq F \cup (C_1 \setminus X_{j,k}) \cup C_2 \cup (C_3 \setminus X_{j,k}) \cup (B_3 \setminus X_{j,k}) \cup A$, and it follows that $|V(H)| \leq h_{j,k}$.

We have now proven (a). It remains to prove (b). Our goal is to define two subsets, D_1 and D_2, of $V(Q_{j,k})$, satisfying the following two properties:

(b.1) $\max\{|D_1|, |D_2|\} = h_{j,k}$;
(b.2) for all $i \in \{1, 2\}$, either $Q[D_i]$ is a 5-hyperhole, or D_i is the union of two (possibly empty) cliques.

Obviously, if both (b.1) and (b.2) hold, then (b) follows immediately.

If $j^* = 1$, then we set

- $D_1 := \{a_1, \dots, a_{\ell_j}\} \cup \{b_{j+1}, \dots, b_{j+p_j}\} \cup (C_1 \setminus X_{j,k}) \cup (C_3 \setminus X_{j,k}) \cup F \cup C_2$;
- $D_2 := \{a_1, \dots, a_{\ell_j}\} \cup \{b_{j+1}, \dots, b_{j+p_j}\} \cup (C_1 \setminus X_{j,k}) \cup (C_3 \setminus X_{j,k}) \cup F \cup (B_3 \setminus X_{j,k})$.

On the other hand, if $j^* = 3$, then we set

- $D_1 := \{a_1, \dots, a_{\ell_j}\} \cup \{b_{j+1}, \dots, b_{j+p_j}\} \cup F \cup (C_1 \setminus X_{j,k}) \cup (C_3 \setminus X_{j,k}) \cup (B_3 \setminus X_{j,k})$;
- $D_2 := A \cup C_2 \cup F \cup (C_1 \setminus X_{j,k}) \cup (C_3 \setminus X_{j,k}) \cup (B_3 \setminus X_{j,k})$.

The fact that (b.1) holds is immediate from the construction. The fact that (b.2) holds follows by routine checking.[63] So, (b) holds. This proves Claim 1. ♦

We now complete our description of the algorithm. First, we find indices $j \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$ for which $\max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\}$ is minimum; clearly, indices j, k can be found in $O(n^3)$ time. We now compute any (j,k)-good set $X_{j,k}$, and we form the graphs $Q[X_{j,k}]$ and $Q_{j,k} := Q \setminus X_{j,k}$; this can be done in further $O(n^2)$ time. By Claim 1, Lemma 6.6, and the choice of j, k, we have that $Q_{j,k}$ is a 5-pseudocrown, $\chi(Q_{j,k}) = \max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\}$, and $\chi(Q) = |B_2| + \chi(Q_{j,k})$.[64]

We can obtain a simplicial elimination ordering of $Q[X_{j,k}]$ by first listing all vertices of $(B_1 \cup B_2 \cup B_3) \cap X_{j,k}$ (in any order), and then listing all vertices of $(C_1 \cup C_3) \cap X_{j,k}$ (in

[63]Here, we simply analyze which (if any) of the sets from the unions used to define D_1 and D_2 are empty.

[64]Let us justify this in a bit more detail. By Lemma 6.6(b), there exist indices $j' \in \{0, \dots, \min\{|B_1|, |B_2|\}\}$ and $k' \in \{0, \dots, \min\{|C_3|, |B_2|\}\}$ such that for any (j', k')-good set $X_{j',k'}$, we have that $\chi(Q) = |B_2| + \chi(Q_{j',k'})$, where $Q_{j',k'} := Q \setminus X_{j',k'}$. Next, by Lemma 6.6(a), we have that $\chi(Q[X_{j,k}]) = |B_2| = \chi(Q[X_{j',k'}])$. Furthermore, by Claim 1, and by the minimality of $\max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\}$, we have that $\chi(Q_{j,k}) = \max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\} \leq \max\{r_{j',k'}, \lceil h_{j',k'}/2 \rceil\} = \chi(Q_{j',k'})$. But now $|B_2| + \chi(Q_{j,k}) \leq |B_2| + \chi(Q_{j',k'}) = \chi(Q) \leq \chi(Q[X_{j,k}]) + \chi(Q_{j,k}) = |B_2| + \chi(Q_{j,k})$, and so $\chi(Q) = |B_2| + \chi(Q_{j,k})$.

any order). We now color $Q[X_{j,k}]$ greedily in $O(n^2)$ time using the reverse of our simplicial elimination ordering; clearly, this coloring is optimal, and so by Lemma 6.6(a), it uses precisely $|B_2|$ colors. Next, we obtain an optimal coloring of the 5-pseudocrown $Q_{j,k}$ by calling the $O(n^3)$ time algorithm from Lemma 6.5; this coloring uses $\chi(Q_{j,k}) = \max\{r_{j,k}, \lceil h_{j,k}/2 \rceil\}$ colors. After possibly renaming colors, we may assume that our colorings of $Q_{j,k}$ and $Q[X_{j,k}]$ use disjoint color sets; we obtain a proper coloring of Q by taking the union of these two colorings. The number of colors used by this coloring of Q is $|B_2| + \chi(Q_{j,k}) = \chi(Q)$, and so the coloring is optimal. We return our coloring of Q, and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. □

6.3 Coloring $(4K_1, C_4, C_6, C_7)$-free graphs

Theorem 6.8. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *Either an optimal coloring of G, or the true statement that G is not $(4K_1, C_4, C_6, C_7)$-free;*
- ***Running time:*** $O(n^3)$.

Proof. We first call the $O(n^3)$ time algorithm from Lemma 3.3 with input G, and we obtain a maximal sequence $v_1, \dots, v_s$ ($s \geq 0$) of pairwise distinct vertices of G such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$.

Suppose first that $V(G) = \{v_1, \dots, v_s\}$. Then $v_1, \dots, v_s$ is a simplicial elimination ordering of G, and we obtain an optimal coloring of G by coloring G greedily using the ordering $v_s, \dots, v_1$; this takes $O(n^2)$ time. We return this coloring, and we stop.

From now on, we assume that $\{v_1, \dots, v_s\} \subsetneqq V(G)$. We now form the graph $H := G \setminus \{v_1, \dots, v_s\}$, and we find the set U of all universal vertices of H; this takes $O(n^2)$ time. Since H has no simplicial vertices (by the maximality of $v_1, \dots, v_s$), we see that H is not complete; consequently, $U \subsetneqq V(H)$. We form the graph $Q := H \setminus U$ in further $O(n^2)$ time, and then using the $O(n^3)$ time algorithms from Lemmas 6.5 and 6.7, we either obtain an optimal coloring of Q, or we determine that Q is neither a 5-crown nor a 5-basket. If Q is neither a 5-crown nor a 5-basket, then we return the answer that G is not $(4K_1, C_4, C_6, C_7)$-free, and we stop; Theorem 3.12 guarantees that this is correct. Assume now that we obtained an optimal coloring of Q. We then extend this coloring to an optimal coloring of H by assigning a new color to each vertex of U (each vertex of U gets a different color, and so we use $|U|$ new colors). Finally, we extend this coloring of H to an optimal coloring of G by greedily assigning colors to vertices $v_s, \dots, v_1$ (in that order); we return the resulting coloring of G, and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. □

7 $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7

Theorem 7.1. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *One of the following:*
  - *The true statement that G is $(4K_1, C_4, C_6)$-free and contains an induced C_7, together with a special partition of G, the list of all maximal cliques of G, and a minimum clique cover of G,*
  - *The true statement that either G is not $(4K_1, C_4, C_6)$-free or G does not contain an induced C_7;*
- ***Running time:*** $O(n^2)$.

Proof. We first call the $O(n^2)$ time algorithm from Lemma 3.5 with input G, and we obtain the partition $\mathcal{P}$ of $V(G)$ into true twin classes of G, as well as the quotient graph $G_{\mathcal{P}}$. Since none of the graphs $4K_1, C_4, C_6, C_7$ has a pair of true twins, the following are equivalent:

(1) G is $(4K_1, C_4, C_6)$-free and contains an induced C_7;
(2) $G_{\mathcal{P}}$ is $(4K_1, C_4, C_6)$-free and contains an induced C_7.

Obviously, $G_{\mathcal{P}}$ does not contain a pair of true twins. So, by Theorem 3.14, if $|\mathcal{P}| \geq 14$, then (2) does not hold; on the other hand, if $|\mathcal{P}| \leq 13$, then we can check whether (2) holds in $O(1)$ time. If (2) does not hold, then we return the statement that either G is not $(4K_1, C_4, C_6)$-free or G does not contain an induced C_7, and we stop. From now on, we assume that (2) holds. By Theorem 3.13, $G_{\mathcal{P}}$ admits a special partition. Since $|V(G_{\mathcal{P}})| \leq 13$, a special partition $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$ of $G_{\mathcal{P}}$ can be found in $O(1)$ time. But then $P := (\bigcup X_0, \dots, \bigcup X_6; \bigcup Y_0, \dots, \bigcup Y_6; \bigcup Z_0, \dots, \bigcup Z_6; \bigcup W)$ is a special partition of G. Further, since $|\mathcal{P}| \leq 13$, a list $C_1, \dots, C_p$ of all maximal cliques of $G_{\mathcal{P}}$ and a minimum clique cover $\{D_1, \dots, D_q\}$ of $G_{\mathcal{P}}$ can both be found in $O(1)$ time. It is then clear that $\bigcup C_1, \dots, \bigcup C_p$ is the list of all maximal cliques of G, and that $\{\bigcup D_1, \dots, \bigcup D_q\}$ is a minimum clique cover of G. We now return the answer that G is $(4K_1, C_4, C_6)$-free and contains an induced C_7, together with the special partition P, the list of maximal cliques $\bigcup C_1, \dots, \bigcup C_p$, and the clique cover $\{\bigcup D_1, \dots, \bigcup D_q\}$, and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^2)$. □

The remainder of this section is organized as follows. In subsection 7.1, we prove some properties of special partitions. In subsection 7.2, we show that each $(4K_1, C_4, C_6)$-free graph G that contains an induced C_7, has at most $\min\{|V(G)|, 9\}$ maximal cliques (see Theorem 7.4). In subsection 7.3, we describe an $O(n^3)$ time coloring algorithm for $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7 (see Theorem 7.8).

7.1 Some properties of special partitions

Lemma 7.2. *Let $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$ be a special partition of a graph G, and set $Y := \bigcup_{i \in \mathbb{Z}_7} Y_i$ and $Z := \bigcup_{i \in \mathbb{Z}_7} Z_i$. Then exactly one of the following holds:*

(a) there exists an index $i \in \mathbb{Z}_7$ such that $Y = Y_i \cup Y_{i+3}$ and $Z = Z_i \cup Z_{i+3} \cup Z_{i+4}$;
(b) there exists an index $i \in \mathbb{Z}_7$ such that all the following hold:
  - *$Y = Y_i$ and $Z = Z_{i+1} \cup Z_{i+2} \cup Z_{i+3}$,*
  - *Y_i, Z_{i+2} are both nonempty,*
  - *at most one of Z_{i+1}, Z_{i+3} is nonempty.*

Moreover, all the following hold:

- *Y and Z are both cliques;*
- *if (a) holds, then $Y \cup Z$ is a clique;*
- *if (b) holds, then $Y \cup Z$ is not a clique.*

Proof. The definition of a special partition readily implies that Y and Z are both cliques. It also guarantees that if (a) holds, then $Y \cup Z$ is a clique. On the other hand, if (b) holds, then $Y \cup Z$ is not a clique (this is because Y_i is anticomplete to Z_{i+2} for all $i \in \mathbb{Z}_7$). It is also obvious that at most one of (a) and (b) holds. It remains to show that at least one of (a) and (b) holds.

Claim 1. There exists an index $i \in \mathbb{Z}_7$ such that $Y = Y_i \cup Y_{i+3}$.

Proof of Claim 1. We may assume that at least one of $Y_0, \dots, Y_6$ is nonempty, for otherwise, the result is immediate. By symmetry, we may assume that $Y_0 \neq \emptyset$. It then follows from the definition of a special partition that Y_1, Y_2, Y_5, Y_6 are all empty, and that at most one of Y_3, Y_4 is nonempty. Consequently, either $Y = Y_0 \cup Y_3$ or $Y = Y_0 \cup Y_4$. In the former case, we set $i = 0$, and in the latter case, we set $i = 4$; then $Y = Y_i \cup Y_{i+3}$, and we are done. This proves Claim 1. ♦

Claim 2. There exists an index $i \in \mathbb{Z}_7$ such that $Z = Z_i \cup Z_{i+3} \cup Z_{i+4}$.

Proof of Claim 2. Suppose first that, for some index $j \in \mathbb{Z}_7$, both Z_j, Z_{j+1} are nonempty; by symmetry, we may assume that Z_3, Z_4 are both nonempty. Since $Z_3 \neq \emptyset$, the definition of a special partition guarantees that Z_1, Z_5 are both empty; similarly, since $Z_4 \neq \emptyset$, we have that Z_2, Z_6 are both empty. It now follows that $Z = Z_0 \cup Z_3 \cup Z_4$, and we are done (with $i = 0$).

Suppose now that for all indices $j \in \mathbb{Z}_7$, at least one of Z_j, Z_{j+1} is empty. We may assume that at least one of $Z_0, \dots, Z_6$ is nonempty, for otherwise, the result is immediate. By symmetry, we may assume that $Z_0 \neq \emptyset$. By our supposition, this implies that Z_1, Z_6 are both empty. Furthermore, by the definition of a special partition, Z_2, Z_5 are both empty. But now $Z = Z_0 \cup Z_3 \cup Z_4$,[65] and again we are done (with $i = 0$). This proves Claim 2. ♦

Claim 3. If either $Y_0, \dots, Y_6$ are all empty, or at least two of $Y_0, \dots, Y_6$ are nonempty, then (a) holds.

Proof of Claim 3. If $Y_0, \dots, Y_6$ are all empty, then the result follows immediately from Claim 2. Suppose now that at least two of $Y_0, \dots, Y_6$ are nonempty. By Claim 1, and by symmetry, we may assume that $Y = Y_0 \cup Y_3$, and that Y_0, Y_3 are both nonempty. Since $Y_0 \neq \emptyset$, it follows from the definition of a special partition that Z_5, Z_6 are both empty. Similarly, since $Y_3 \neq \emptyset$, we have that Z_1, Z_2 are both empty. Thus, $Z = Z_0 \cup Z_3 \cup Z_4$. But now (a) holds for $i = 0$. This proves Claim 3. ♦

In view of Claim 3, we may assume from now on that exactly one of $Y_0, \dots, Y_6$ is nonempty. By symmetry, we may assume that $Y_0 \neq \emptyset$, and that $Y_1, \dots, Y_6$ are all empty; in particular, $Y = Y_0$. By the definition of a special partition, it follows that Z_5, Z_6 are both empty.

Suppose first that $Z_2 \neq \emptyset$. It then follows from the definition of a special partition that Z_0, Z_4 are both empty. We now have that $Z = Z_1 \cup Z_2 \cup Z_3$. But by the definition of a special partition, we know that at most one of Z_1, Z_3 is nonempty, and we deduce that (b) holds (for $i = 0$).

Suppose now that $Z_2 = \emptyset$. Then $Z = Z_0 \cup Z_1 \cup Z_3 \cup Z_4$. By the definition of special partition, at most one of Z_1, Z_3 is nonempty. So, either $Z = Z_0 \cup Z_1 \cup Z_4$ or $Z = Z_0 \cup Z_3 \cup Z_4$. In the former case, we set $i = 4$, and in the latter case, we set $i = 0$. Now (a) holds. □

Lemma 7.3. *Let G be a graph that admits a special partition $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$, and set $Y := \bigcup_{i \in \mathbb{Z}_7} Y_i$ and $Z := \bigcup_{i \in \mathbb{Z}_7} Z_i$. Then all the following hold:*

(a) G is $(4K_1, C_4, C_6)$-free and contains an induced C_7;
(b) for all indices $i \in \mathbb{Z}_7$, the graph $G \setminus X_i$ is C_7-free;
(c) there exists an index $i \in \mathbb{Z}_7$ such that X_i is complete to $Y \cup Z$;
(d) for all indices $i \in \mathbb{Z}_7$ such that X_i is complete to $Y \cup Z$, the graph $G \setminus X_i$ is 5-pyramid-free.[66]

Proof. Set $X := \bigcup_{i \in \mathbb{Z}_7} X_i$.

Claim 1. No induced subgraph of G that is isomoprhic to one of $4K_1, C_6, C_7$ contains a vertex of $Y \cup Z \cup W$.

Proof of Claim 1. By the definition of a special partition, all vertices of W are universal in G; since none of $4K_1, C_6, C_7$ contains a universal vertex, it follows that no induced subgraph of G that is isomorphic to one of $4K_1, C_6, C_7$ contains a vertex of W.

Next, we show that no induced subgraph of G that is isomorphic to one of $4K_1, C_6, C_7$ contains a vertex of Z. By symmetry, it suffices to show that no such induced subgraph of G contains a vertex of Z_2. By the definition of a special partition, for all $z_2 \in Z_2$, we have that $V(G) \setminus N_G[z_2] = X_0 \cup X_1 \cup Y_0$. But note that $X_0 \cup X_1 \cup Y_0$ is a clique of G. Since none of

[65] We also have that at least one of Z_3, Z_4 is empty, but we do not need this fact.

[66] We remind the reader that the 5-pyramid was defined in subsection 3.4 (see Figure 3.2).

$4K_1, C_6, C_7$ contains a vertex whose nonneighborhood is a clique, we deduce that no induced subgraph of G isomorphic to one of $4K_1, C_6, C_7$ contains a vertex of Z_2.

It remains to show that no induced subgraph of $H := G \setminus (Z \cup W)$ that is isomorphic to one of $4K_1, C_6, C_7$ contains a vertex of Y. By symmetry, it suffices to show that no such induced subgraph of H contains a vertex of Y_0. By the definition of a special partition, for all $y_0 \in Y_0$, we have that $N_H[y_0] = X_0 \cup X_1 \cup X_4 \cup Y$ and $V(H) \setminus N_H[y_0] = X_2 \cup X_3 \cup X_5 \cup X_6$. But note that $X_2 \cup X_3$ and $X_5 \cup X_6$ are cliques, anticomplete to each other. Since none of $4K_1, C_6, C_7$ has a vertex whose nonneighborhood can be partitioned into two (possibly empty) cliques, anticomplete to each other, we deduce that no induced subgraph of H isomorphic to one of $4K_1, C_6, C_7$ contains a vertex of Y_0. This proves Claim 1. ♦

Claim 2. G is $(4K_1, C_4, C_6)$-free.

Proof of Claim 2. Since $G[X]$ is a 7-hyperhole, it is clear that it is $(4K_1, C_6)$-free. It then follows from Claim 1 that G is $(4K_1, C_6)$-free. It remains to show that G is C_4-free. Suppose otherwise, and let c_0, c_1, c_2, c_3, c_0 be a 4-hole in G. Set $C := \{c_0, c_1, c_2, c_3\}$.

Suppose that $C \cap (Y \cup Z \cup W)$ is a clique. Since c_0, c_1, c_2, c_3, c_0 is a 4-hole, it follows that $C \cap X$ is not a stable set; by symmetry, we may assume that $c_0, c_1 \in C \cap X$. We may further assume by symmetry that either $c_0, c_1 \in X_0$, or $c_0 \in X_0$ and $c_1 \in X_1$. However, the former is impossible because any two vertices of X_0 are true twins in G, and c_0, c_1 are not true twins. So, $c_0 \in X_0$ and $c_1 \in X_1$. It then follows from the definition of a special partition that $N_G(c_0) \setminus N_G[c_1] = X_6 \cup Y_3 \cup Y_6 \cup Z_3$ and $N_G(c_1) \setminus N_G[c_0] = X_2 \cup Y_1 \cup Y_4 \cup Z_1$, and consequently, that $N_G(c_0) \setminus N_G[c_1]$ is anticomplete to $N_G(c_1) \setminus N_G[c_0]$. But this is impossible since $c_3 \in N_G(c_0) \setminus N_G[c_1]$ and $c_2 \in N_G(c_1) \setminus N_G[c_0]$ are adjacent.

We have now shown that $C \cap (Y \cup Z \cup W)$ is not a clique; in particular, $Y \cup Z \cup W$ is not a clique. Since all vertices in W are universal in G, it follows that $Y \cup Z$ is not a clique. So, by Lemma 7.2, and by symmetry, we may assume that $Y = Y_0$ and $Z = Z_1 \cup Z_2 \cup Z_3$, that Y_0, Z_2 are both nonempty, and that at most one of Z_1, Z_3 is nonempty. But then all nonedges in $G[Y \cup Z \cup W]$ are between Y_0 and Z_2. Since $C \cap (Y \cup Z \cup W)$ is not a clique, and since c_0, c_1, c_2, c_3, c_0 is a 4-hole, we may assume by symmetry that $c_0 \in Y_0$ and $c_2 \in Z_2$. But then $N_G(c_0) \cap N_G(c_2) = X_4 \cup Z_1 \cup Z_3 \cup W$; so, $c_1, c_3 \in X_4 \cup Z_1 \cup Z_3 \cup W$. But this is impossible since c_1, c_3 are nonadjacent, and $X_4 \cup Z_1 \cup Z_3 \cup W$ is a clique. This proves Claim 2. ♦

Claim 3. G contains an induced C_7. Furthermore, for all $i \in \mathbb{Z}_7$, the graph $G \setminus X_i$ is C_7-free.

Proof of Claim 3. Since G contains a 7-hyperhole (namely, $G[X]$), it is clear that G contains an induced C_7. Furthermore, by Claim 1, all 7-holes of G are in fact 7-holes of $G[X]$, and clearly, for all $i \in \mathbb{Z}_7$, the graph $G[X \setminus X_i]$ is chordal,[67] and therefore C_7-free. Thus, for all $i \in \mathbb{Z}_7$, the graph $G \setminus X_i$ is C_7-free. This proves Claim 3. ♦

Claim 4. There exists an index $i \in \mathbb{Z}_7$ such that X_i is complete to $Y \cup Z$.

Proof of Claim 4. By Lemma 7.2, and by symmetry, we may assume that either $Y = Y_0 \cup Y_3$ and $Z = Z_0 \cup Z_3 \cup Z_4$, or $Y = Y_0$ and $Z = Z_1 \cup Z_2 \cup Z_3$. In either case, X_4 is complete to $Y \cup Z$. This proves Claim 4. ♦

Claim 5. If P is an induced 5-pyramid in G, then there exists some index $i \in \mathbb{Z}_7$ such that $V(P)$ intersects each of X_{i+4}, Y_i, Z_{i+2}.

Proof of Claim 5. Assume that P is an induced 5-pyramid in G; we must show that there exists some index $i \in \mathbb{Z}_7$ such that $V(P)$ intersects each of X_{i+4}, Y_i, Z_{i+2}.

[67] This is obvious, but it also follows from Lemma 3.7(c).

First, we may assume that $Y \cup Z \cup W \subseteq V(P)$, for otherwise, we consider the graph $G[X \cup V(P)] = G \setminus \Big((Y \cup Z \cup W) \setminus V(P)\Big)$ instead of G. Since every vertex in W is universal in G, and since the 5-pyramid contains no universal vertices, it follows that $W = \emptyset$. Further, note that any two vertices that belong to the same set of our special partition are true twins in G. Since the 5-pyramid contains no pair of true twins, we may now assume that $X_0, \dots, X_6$ are all singletons, and that none of $Y_0, \dots, Y_6, Z_0, \dots, Z_6$ has more than one vertex.[68] For all $i \in \mathbb{Z}_7$, we set $X_i = \{x_i\}$; then $G[X]$ is a 7-hole of the form $x_0, \dots, x_6, x_0$.

Suppose first that $Y \cup Z$ is a clique. Now, note that the deletion of a clique from the 5-pyramid always yields a graph that contains at least one of $K_3, K_{1,3}, C_5$ as an induced subgraph;[69] consequently, $P \setminus (Y \cup Z)$ is not $(K_3, K_{1,3}, C_5)$-free. But on the other hand, $P \setminus (Y \cup Z)$ is an induced subgraph of the 7-hole $G[X]$, which is obviously $(K_3, K_{1,3}, C_5)$-free, a contradiction.

Thus, $Y \cup Z$ is not a clique. By Lemma 7.2, and by symmetry, we may now assume that $Y = Y_0$ and $Z = Z_1 \cup Z_2 \cup Z_3$, that both Y_0 and Z_2 are nonempty, and that at most one of Z_1, Z_3 is nonempty. Since Y_0, Z_2 are both nonempty, we see that they are both singletons, and we set $Y_0 = \{y_0\}$ and $Z_2 = \{z_2\}$. Now y_0, z_2 are nonadjacent vertices of the 5-pyramid P, and so they have a (unique) common neighbor in P, call it p. But note that $N_G(y_0) \cap N_G(z_2) = X_4 \cup Z_1 \cup Z_3$; so, $p \in X_4 \cup Z_1 \cup Z_3$. Suppose first that $p \in Z_1 \cup Z_3$. Then z_2, p are adjacent vertices of the 5-pyramid P, and consequently, some two vertices of $V(P) \setminus \{z_2, p\}$ are anticomplete to $\{z_2, p\}$ in P (and therefore, in G as well). But if $p \in Z_1$, then x_0 is the unique common nonneighbor of z_2, p in G; and if $p \in Z_3$, then x_1 is the unique common nonneighbor of z_2, p in G. So, $p \notin Z_1 \cup Z_3$, and we deduce that $p \in X_4$. But now $V(P)$ intersects each of X_4, Y_0, Z_2, and we are done. This proves Claim 5. ♦

> **Claim 6.** For all indices $i \in \mathbb{Z}_7$ such that X_i is complete to $Y \cup Z$, the graph $G \setminus X_i$ is 5-pyramid-free.

Proof of Claim 6. Fix $i \in \mathbb{Z}_7$ such that X_i is complete to $Y \cup Z$. We may assume that G contains an induced 5-pyramid, for otherwise, we are done. By Claim 5, and by symmetry, we may assume that Y_0, Z_2 are both nonempty; it then follows from Lemma 7.2 that $Y = Y_0$ and $Z = Z_1 \cup Z_2 \cup Z_3$, and that at least one of Z_1, Z_3 is empty. So, by Claim 5, every induced 5-pyramid of G intersects each of X_4, Y_0, Z_2. In particular, $G \setminus X_4$ is 5-pyramid-free. On the other hand, since X_i is complete to $Y_0 \cup Z_2$, and since Y_0, Z_2 are both nonempty, the definition of a special partition guarantees that $i = 4$. So, $G \setminus X_i$ is 5-pyramid-free. This proves Claim 6. ♦

By Claims 2 and 3, we see that (a) and (b) hold. By Claim 4, (c) holds. By Claim 6, (d) holds. This completes the argument. □

7.2 Maximal cliques in $(4K_1, C_4, C_6)$-free graph that contain an induced C_7

Theorem 7.4. *Let G be a $(4K_1, C_4, C_6)$-free graph that contains an induced C_7. Then the number of maximal cliques of G is at most* $\min\{|V(G)|, 9\}$.

Proof. We may assume inductively that all $(4K_1, C_4, C_6)$-free graphs G' that contain an induced C_7, and that have fewer than $|V(G)|$ vertices, have at most $\min\{|V(G')|, 9\}$ maximal cliques.

[68] Let us justify this in a bit more detail. For all $i \in \mathbb{Z}_7$, if $X_i \cap V(P) \neq \emptyset$, then we let x_i be the unique vertex of $X_i \cap V(P)$, and otherwise, we let x_i be any vertex of $V(P)$. Further, for all $i \in \mathbb{Z}_i$, we set $X'_i := \{x_i\}$, $Y'_i := Y_i \cap V(P)$, and $Z'_i := Z_i \cap V(P)$. Finally, we set $G' := G[\bigcup_{i \in \mathbb{Z}_i}(X'_i \cup Y'_i \cup Z'_i)]$. If necessary, we may now consider the graph G' with the associated special partition $(X'_0, \dots, X'_6; Y'_0, \dots, Y'_6; Z'_0, \dots, Z'_6; \emptyset)$, instead of the graph G with the associated special partition $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$.

[69] As usual, for positive integers p and q, we denote by $K_{p,q}$ the graph whose vertex set can be partitioned into two stable sets, one of size p and the other one of size q, that are complete to each other.

Let $\mathcal{P}$ be the partition of $V(G)$ into true twin classes. Since G is $(4K_1, C_4, C_6)$-free and contains an induced C_7, the same holds for its quotient graph $G_{\mathcal{P}}$.[70] Next, note that the maximal cliques of G are precisely the sets of the form $\bigcup C$, where C is a maximal clique of the quotient graph $G_{\mathcal{P}}$. So, the number of maximal cliques of G is equal to the number of maximal cliques of $G_{\mathcal{P}}$. Thus, if G has a pair of true twins, then the result follows from the induction hypothesis. From now on, we assume that G contains no pair of true twins.

By Theorem 3.13, G admits a special partition $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$. Note that the maximal cliques of G are precisely the sets of the form $C \cup W$, where C is a maximal clique of $G \setminus W$;[71] so, the number of maximal cliques of G is the same as the number of maximal cliques of $G \setminus W$. Thus, if $W \neq \emptyset$, the result follows from the induction hypothesis. From now on, we assume that $W = \emptyset$.

Note that each of the sets X_i, Y_i, Z_i ($i \in \mathbb{Z}_7$) is either empty or a true twin class in G. Since G contains no pair of true twins, and since the X_i's are nonempty, we have that sets $X_0, \dots, X_6$ are all singletons, and each of the sets $Y_0, \dots, Y_6, Z_0, \dots, Z_6$ is either empty or a singleton. For all $i \in \mathbb{Z}_7$, we set $X_i = \{x_i\}$. Further, we set $X := \bigcup_{i \in \mathbb{Z}_7} X_i$, $Y := \bigcup_{i \in \mathbb{Z}_7} Y_i$, and $Z := \bigcup_{i \in \mathbb{Z}_7} Z_i$. Note that $X = \{x_0, \dots, x_6\}$, and furthermore, $x_0, \dots, x_6, x_0$ is a 7-hole in G.

If $Y \cup Z = \emptyset$, then $|V(G)| = 7$, and the maximal cliques of G are the cliques of the form $\{x_i, x_{i+1}\}$, for $i \in \mathbb{Z}_7$; there are precisely seven such cliques, and we are done.

Suppose now that $|Y \cup Z| = 1$; then $|V(G)| = 8$, and it suffices to show that G has at most eight maximal cliques. By symmetry, there are two cases two consider: when $|Y| = 1$ and $Z = \emptyset$, and when $Y = \emptyset$ and $|Z| = 1$. Suppose first that $|Y| = 1$ and $Z = \emptyset$. By symmetry, we may assume that $Y = Y_0$; let y_0 be the unique vertex of Y_0. Then the maximal cliques of G are precisely the following: $\{x_0, x_1, y_0\}$, $\{x_4, y_0\}$, and all the cliques of the form $\{x_i, x_{i+1}\}$ with $i \in \mathbb{Z}_7 \setminus \{0\}$; there are precisely eight such cliques, and we are done. Suppose now that $Y = \emptyset$ and $|Z| = 1$. By symmetry, we may assume that $Z = Z_0$; let z_0 be the unique vertex of Z_0. Then the maximal cliques of G are precisely the following: $\{x_i, x_{i+1}, z_0\}$ for $i \in \{0, 1, 2, 3\}$, and $\{x_i, x_{i+1}\}$ for $i \in \{4, 5, 6\}$; there are precisely seven such cliques, and we are done.

From now on, we assume that $|Y \cup Z| \geq 2$. Then $|V(G)| \geq 9$, and we need to show that G has at most nine maximal cliques.

To simplify notation, for all $i \in \mathbb{Z}_7$, we set $D_i := N_G[x_i] \setminus \{x_{i-1}, x_{i+1}\}$, and we note that $D_i = \{x_i\} \cup \Big(N_G(x_i) \cap (Y \cup Z)\Big)$.

> **Claim 1.** For all maximal cliques K of G, there exists an index $i \in \mathbb{Z}_7$ such that either $K \subseteq D_i$ or $K = N_G[x_i] \cap N_G[x_{i+1}]$.

Proof of Claim 1. Fix a maximal clique K of G. Lemma 7.3(c), then guarantees that $K \cap X \neq \emptyset$. Since $G[X]$ is a 7-hole of the form $x_0, x_1, \dots, x_6, x_0$, we may assume by symmetry that either $K \cap X = \{x_0\}$ or $K \cap X = \{x_0, x_1\}$. In the former case, we have that $K \subseteq D_0$, and we are done. So, suppose that $K \cap X = \{x_0, x_1\}$. Then $K \subseteq N_G[x_0] \cap N_G[x_1]$. But by the definition of a special partition, and by the fact that $W = \emptyset$, we have that $N_G[x_0] \cap N_G[x_1] = \{x_0, x_1\} \cup Y_0 \cup Z_0 \cup Z_4 \cup Z_5 \cup Z_6$ is a clique. So, by the maximality of K, we have that $K = N_G[x_0] \cap N_G[x_1]$. This proves Claim 1. ♦

Recall that $|V(G)| \geq 9$. In view of Claim 1, it suffices to show that at most two maximal cliques of G are included in one of $D_0, \dots, D_6$.

By Lemma 7.2, and by symmetry, we may assume that one of the following holds:

(a) $Y = Y_0 \cup Y_3$ and $Z = Z_0 \cup Z_3 \cup Z_4$;

[70] Indeed, $G_{\mathcal{P}}$ is (isomorphic to) an induced subgraph of G, and so since G is $(4K_1, C_4, C_6)$-free, so is $G_{\mathcal{P}}$. On the other hand, since C_7 does not contain a pair of true twins, the fact that G contains an induced C_7 implies that $G_{\mathcal{P}}$ does as well.

[71] This is because all vertices of W are universal in G.

(b) all the following hold:
   - $Y = Y_0$ and $Z = Z_1 \cup Z_2 \cup Z_3$,
   - Y_0, Z_2 are both nonempty,
   - at most one of Z_1, Z_3 is nonempty.

Suppose first that (a) holds. Then by Lemma 7.2, $Y \cup Z$ is a clique. Consequently, $D_0, \dots, D_6$ are all cliques, and it suffices to show that at most two of them are maximal in G. Now, we have the following:

- $D_0 = \{x_0\} \cup Y_0 \cup Y_3 \cup Z_0 \cup Z_3 \cup Z_4$;
- $D_1 = \{x_1\} \cup Y_0 \cup Z_0 \cup Z_4$;
- $D_2 = \{x_2\} \cup Z_0$;
- $D_3 = \{x_3\} \cup Y_3 \cup Z_0 \cup Z_3$;
- $D_4 = \{x_4\} \cup Y_0 \cup Y_3 \cup Z_0 \cup Z_3 \cup Z_4$;
- $D_5 = \{x_5\} \cup Z_3 \cup Z_4$;
- $D_6 = \{x_6\} \cup Z_3 \cup Z_4$.

Now, note that D_1 is a proper subset of the clique $\{x_0, x_1\} \cup Y_0 \cup Z_0 \cup Z_4$; that D_2 is a proper subset of the clique $\{x_2, x_3\} \cup Z_0$; that D_3 is a proper subset of the clique $\{x_3, x_4\} \cup Y_3 \cup Z_0 \cup Z_3$; and that D_5, D_6 are both proper subsets of the clique $\{x_5, x_6\} \cup Z_3 \cup Z_4$. Thus, none of D_1, D_2, D_3, D_5, D_6 is a maximal clique of G; consequently, at most two of the cliques $D_0, \dots, D_6$ are maximal cliques of G, and we are done.

Suppose now that (b) holds. It then follows from the definition of special partition that Y and Z are cliques, and that $Y = Y_0$ is complete to $Z_1 \cup Z_3 = Z \setminus Z_2$ and anticomplete to Z_2. Furthermore, we have the following:

- $D_0 = \{x_0\} \cup Y_0 \cup Z_3$;
- $D_1 = \{x_1\} \cup Y_0 \cup Z_1$;
- $D_2 = \{x_2\} \cup Z_1 \cup Z_2$;
- $D_3 = \{x_3\} \cup Z_1 \cup Z_2 \cup Z_3$;
- $D_4 = \{x_4\} \cup Y_0 \cup Z_1 \cup Z_2 \cup Z_3$;
- $D_5 = \{x_5\} \cup Z_1 \cup Z_2 \cup Z_3$;
- $D_6 = \{x_6\} \cup Z_2 \cup Z_3$.

Note that D_4 is not a clique, and that the remaining D_i's are all cliques. Further, $G[D_4]$ has exactly two maximal cliques, namely $D_4^Y := \{x_4\} \cup Y_0 \cup Z_1 \cup Z_3$ and $D_4^Z := \{x_4\} \cup Z_1 \cup Z_2 \cup Z_3$. It now suffices to show that at most two of the cliques $D_0, D_1, D_2, D_3, D_4^Y, D_4^Z, D_5, D_6$ are maximal in G.

Note that D_2 is a proper subset of the clique $\{x_2, x_3\} \cup Z_1 \cup Z_2$; that D_3 is a proper subset of the clique $\{x_3, x_4\} \cup Z_1 \cup Z_2 \cup Z_3$; that D_4^Z, D_5 are both proper subsets of the clique $\{x_4, x_5\} \cup Z_1 \cup Z_2 \cup Z_3$; and that D_6 is a proper subset of the clique $\{x_5, x_6\} \cup Z_2 \cup Z_3$. Thus, none of $D_2, D_3, D_4^Z, D_5, D_6$ is a maximal clique of G. Further, since at most one of Z_1, Z_3 is nonempty, we see that $D_0 = \{x_0\} \cup Y_0$ or $D_1 = \{x_1\} \cup Y_0$. Since both $\{x_0\} \cup Y_0$ and $\{x_1\} \cup Y_0$ are proper subsets of the clique $\{x_0, x_1\} \cup Y_0$, we deduce that at most one of the cliques D_0, D_1 is maximal in G. We have now shown that at most two of the cliques $D_0, D_1, D_2, D_3, D_4^Y, D_4^Z, D_5, D_6$ are maximal in G, and we are done. □

7.3 Coloring $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7

For a graph G, we let $\mathscr{F}(G)$ be the set of all ordered pairs (A, B) that have the following three properties:

- A, B are disjoint subsets of $V(G)$, complete to each other in G;
- $A \neq \emptyset$ and $G[A]$ is a 5-hyperhole;
- B is a (possibly empty) clique of G.

Lemma 7.5. *Let G be a $(4K_1, C_4, C_6, C_7,$ 5-pyramid$)$-free graph. Then* $\chi(G) = \max\Big(\{\omega(G)\} \cup \{\lceil |A|/2 \rceil + |B| \mid (A, B) \in \mathscr{F}(G)\}\Big)$.

Proof. First, it is obvious that $\chi(G) \geq \omega(G)$, and that for all $(A, B) \in \mathscr{F}(G)$, we have that

$$\begin{array}{rcll} \chi(G) & \geq & \chi(G[A \cup B]) & \\ & = & \chi(G[A]) + |B| & \text{because } A \text{ is complete to } B \text{, and } B \text{ is a clique} \\ & \geq & \lceil |A|/\alpha(G[A]) \rceil + |B| & \\ & = & \lceil |A|/2 \rceil + |B| & \text{because } G[A] \text{ is a 5-hyperhole.} \end{array}$$

Thus, $\chi(G) \geq \max\Big(\{\omega(G)\} \cup \{\lceil |A|/2 \rceil + |B| \mid (A, B) \in \mathscr{F}(G)\}\Big)$.

It remains to prove the reverse inequality. Let $v_1, \dots, v_s$ ($s \geq 0$) be a maximal sequence of pairwise distinct vertices of G such that for all $i \in \{1, \dots, s\}$, v_i is simplicial in the graph $G \setminus \{v_1, \dots, v_{i-1}\}$.[72]

Suppose first that $V(G) = \{v_1, \dots, v_s\}$. Then $v_1, \dots, v_s$ is a simplicial elimination ordering of G, and so by Theorem 3.2, G is chordal. Theorem 3.1 then guarantees that G is perfect, and it follows that $\chi(G) = \omega(G) \leq \max\Big(\{\omega(G)\} \cup \{\lceil |A|/2 \rceil + |B| \mid (A, B) \in \mathscr{F}(G)\}\Big)$.

From now on, we assume that $\{v_1, \dots, v_s\} \subsetneqq V(G)$. Set $G' := G \setminus \{v_1, \dots, v_s\}$, and let U be the set of all universal vertices of G'. By the maximality of $v_1, \dots, v_s$, we see that G' contains no simplicial vertices, and in particular, G' is not complete; so, $U \subsetneqq V(G')$. Set $Q := V(G') \setminus U$. Now, since G is $(4K_1, C_4, C_6, C_7, \text{5-pyramid})$-free, so is G'. We already saw that G' has no simplicial vertices, and so Theorem 3.11 guarantees that Q is a 5-crown. In particular, Q is a 5-ring, and so Theorem 3.8 guarantees that $\chi(Q) = \max\Big(\{\omega(Q)\} \cup \{\lceil |V(H)|/2 \rceil \mid H \text{ is a 5-hyperhole in } Q\}\Big)$.

Now, note that $\omega(G') = \omega(Q) + |U|$ and $\chi(G') = \chi(Q) + |U|$, and that for all hyperholes H in Q, we have that $(V(H), U) \in \mathscr{F}(G)$. So, we have the following:

$$\begin{array}{rcl} \chi(G') & = & \chi(Q) + |U| \\ & = & \max\Big(\{\omega(Q) + |U|\} \cup \{\lceil |V(H)|/2 \rceil + |U| \mid H \text{ is a 5-hyperhole in } Q\}\Big) \\ & \leq & \max\Big(\{\omega(G')\} \cup \{\lceil |A|/2 \rceil + |B| \mid (A, B) \in \mathscr{F}(G)\}\Big) \\ & \leq & \max\Big(\{\omega(G)\} \cup \{\lceil |A|/2 \rceil + |B| \mid (A, B) \in \mathscr{F}(G)\}\Big). \end{array}$$

Finally, we can extend any optimal coloring of G' to a proper coloring of G by assigning colors greedily to vertices $v_s, \dots, v_1$ (in that order); the number of colors used is at most $\max\{\chi(G'), \omega(G)\}$, and so $\chi(G) \leq \max\Big(\{\omega(G)\} \cup \{\lceil |A|/2 \rceil + |B| \mid (A, B) \in \mathscr{F}(G)\}\Big)$. □

Our next goal is to prove an analog of Lemma 7.5 for weighted graphs (see Lemma 7.6 below). First, we need some definitions.

A *weighted graph* is an ordered pair (G, w), where G is graph, and $w : V(G) \to \mathbb{N}$ is a function, called a *weight function*. The *weight* of a vertex $v \in V(G)$ is the number $w(v)$, and the *weight* of a set $S \subseteq V(G)$ is the sum of weights of its vertices, *i.e.* $w(S) = \sum_{v \in S} w(v)$. The *clique number* of (G, w), denoted by $\omega(G, w)$, is the maximum weight of a clique of G, *i.e.* $\omega(G, w) = \max\{w(C) \mid C \text{ is a clique of } G\}$. A *weighted coloring* (G, w) is an indexed set $\{S_1, \dots, S_k\}$ ($k \geq 0$) of (not necessarily distinct) stable sets of G such that for all $v \in V(G)$, at least $w(v)$ of the sets $S_1, \dots, S_k$ contain v. The *chromatic number* of (G, w), denoted by $\chi(G, w)$, is the smallest number of stable sets in any weighted coloring of (G, w). It is clear that $\omega(G, w) \leq \chi(G, w)$. Furthermore, if $w(v) = 0$ for all $v \in V(G)$, then both $\omega(G, w)$ and $\chi(G, w)$ are zero.

[72] If $s = 0$, then the sequence $v_1, \dots, v_s$ is empty and G has no simplicial vertices.

Clearly, if (G, w) is a weighted graph, and H is an induced subgraph of G, then $w \upharpoonright V(H)$ is a weight function for H, and $(H, w \upharpoonright V(H))$ is a weighted graph.[73] To simplify notation, we write (H, w) instead of $(H, w \upharpoonright V(H))$.

Given a weighted graph (G, w) such that w assigns positive weight to at least one vertex of G, we define the graph G^w as follows. The vertex set of G^w can be partitioned into a family $\{C_v\}_{v \in V(G)}$ of (possibly empty) cliques such that for all $v \in V(G)$, we have that $|C_v| = w(v)$, and for all distinct $v_1, v_2 \in V(G)$, C_{v_1} is complete to C_{v_2} in G^w if $v_1v_2 \in E(G)$, and C_{v_1} is anticomplete to C_{v_2} in G^w if $v_1v_2 \notin E(G)$. In other words, G^w is the graph obtained from G by first deleting all vertices of weight zero, and then "blowing up" each remaining vertex v to a clique of size $w(v)$.[74] It is easy to see that $\omega(G, w) = \omega(G^w)$ and $\chi(G, w) = \chi(G^w)$.

Lemma 7.6. *Let (G, w) be a weighted graph. Assume that the weight function w assigns positive weight to at least one vertex of G,[75] and that the graph G^w is $(4K_1, C_4, C_6, C_7$, 5-pyramid$)$-free.[76] Then $\chi(G, w) = \max\Big(\{\omega(G, w)\} \cup \{\lceil w(A)/2 \rceil + w(B) \mid (A, B) \in \mathscr{F}(G)\}\Big)$.*

Proof. In what follows, we let $\{C_v\}_{v \in V(G)}$ be as in the definition of G^w.

First, it is clear that $\chi(G, w) \geq \omega(G, w)$. Further, fix $(A_0, B_0) \in \mathscr{F}(G)$. If $w(A_0) = 0$, then, since B_0 is a clique, we have that $\omega(G, w) \geq w(B_0)$, and so $\chi(G, w) \geq \omega(G, w) \geq w(B_0) = \lceil w(A_0)/2 \rceil + w(B_0)$. Suppose now that $w(A_0) > 0$. Then either $\bigcup_{v \in A_0} C_v$ is the union of two (possibly empty) cliques of G^w, or $G^w[\bigcup_{v \in A_0} C_v]$ is a 5-hyperhole; in either case, we have that $1 \leq \alpha(G^w[\bigcup_{v \in A_0} C_v]) \leq 2$, and we compute:

$$\begin{array}{rcll}
\chi(G) & \geq & \chi(G[A_0 \cup B_0], w) & \\
& = & \chi(G[A_0], w) + w(B_0) & \text{because } A_0 \text{ is complete to } B_0 \text{, and } B_0 \text{ is a clique} \\
& = & \chi(G^w[\bigcup_{v \in A_0} C_v]) + w(B_0) & \\
& \geq & \left\lceil \frac{|\bigcup_{v \in A_0} C_v|}{\alpha(G^w[\bigcup_{v \in A_0} C_v])} \right\rceil + w(B_0) & \\
& \geq & \left\lceil \frac{\sum_{v \in A_0} w(v)}{2} \right\rceil + w(B_0) & \text{because } 1 \leq \alpha(G^w[\bigcup_{v \in A_0} C_v]) \leq 2 \\
& = & \lceil w(A_0)/2 \rceil + w(B_0). &
\end{array}$$

This proves that $\chi(G, w) \geq \max\Big(\{\omega(G, w)\} \cup \{\lceil w(A)/2 \rceil + w(B) \mid (A, B) \in \mathscr{F}(G)\}\Big)$.

It remains to prove the reverse inequality. Clearly, $\chi(G, w) = \chi(G^w)$ and $\omega(G, w) = \omega(G^w)$. If $\chi(G^w) = \omega(G^w)$, then $\chi(G, w) = \chi(G^w) = \omega(G^w) = \omega(G, w)$, and we are done. Suppose now that $\chi(G^w) \neq \omega(G^w)$. Then Lemma 7.5 guarantees that there exists some $(A_0, B_0) \in \mathscr{F}(G^w)$ such that $\chi(G^w) = \lceil |A_0|/2 \rceil + |B_0|$. Let $A_0^* = \{v \in V(G) \mid A_0 \cap C_v \neq \emptyset\}$ and $B_0^* = \{v \in V(G) \mid B_0 \cap C_v \neq \emptyset\}$. It is then clear that $|A_0| \leq w(A_0^*)$ and $|B_0| \leq w(B_0^*)$, as well as that $(A_0^*, B_0^*) \in \mathscr{F}(G)$. We now deduce that $\chi(G, w) = \chi(G^w) = \lceil |A_0|/2 \rceil + |B_0| \leq \lceil w(A_0^*)/2 \rceil + w(B_0^*) \leq \max\{\lceil w(A)/2 \rceil + w(B) \mid (A, B) \in \mathscr{F}(G)\}$, and again we are done. □

Lemma 7.7. *Let $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$ be a special partition of a graph G, and set $Y := \bigcup_{i \in \mathbb{Z}_7} Y_i$ and $Z := \bigcup_{i \in \mathbb{Z}_7} Z_i$. Assume that X_0 is complete to $Y \cup Z$.[77] For indices $j \in \{0, \dots, \min\{|X_0|, |X_2|\}\}$ and $k \in \{0, \dots, \min\{|X_0|, |X_4|\}\}$, a set $A_{j,k} \subseteq V(G)$ is said to be* (j, k)-good *if it satisfies all the following (see Figure 7.1):*

- $X_0 \subseteq A_{j,k} \subseteq X_0 \cup X_2 \cup X_3 \cup X_4 \cup X_5$;

[73]As usual, $w \upharpoonright V(H)$ denotes the restriction of w to $V(H)$.

[74]Since our graphs are nonnull, if w assigns weight zero to each vertex of G, then the graph G^w is undefined.

[75]So, G^w is defined.

[76]It is possible that G itself is not $(4K_1, C_4, C_6, C_7$, 5-pyramid$)$-free. Note, however, that the graph $G \setminus \{v \in V(G) \mid w(v) = 0\}$ is $(4K_1, C_4, C_6, C_7$, 5-pyramid$)$-free.

[77]Since every vertex in W is universal in G, it follows that X_0 is in fact complete to $X \cup Y \cup W$.

- $|X_2 \cap A_{j,k}| = j$ *and* $|X_4 \cap A_{j,k}| = k$*;*
- $|X_3 \cap A_{j,k}| = \min\{|X_3|, |X_0| - j, |X_0| - k\}$*;*
- $|X_5 \cap A_{j,k}| = \min\{|X_5|, |X_0| - k\}$*.*

Then both the following hold:

(a) for all indices $j \in \{0, \dots, \min\{|X_0|, |X_2|\}\}$ *and* $k \in \{0, \dots, \min\{|X_0|, |X_4|\}\}$*, and all* (j,k)*-good sets* $A_{j,k}$*, the graph* $G[A_{j,k}]$ *is chordal and satisfies* $\chi(G[A_{j,k}]) = \omega(G[A_{j,k}]) = |X_0|$*, and the graph* $G \setminus A_{j,k}$ *is* $(4K_1, C_4, C_6, C_7,$ *5-pyramid)-free;*

(b) there exist indices $j \in \{0, \dots, \min\{|X_0|, |X_2|\}\}$ *and* $k \in \{0, \dots, \min\{|X_0|, |X_4|\}\}$ *such that all* (j,k)*-good sets* $A_{j,k}$ *satisfy* $\chi(G) = \chi(G[A_{j,k}]) + \chi(G \setminus A_{j,k})$*.*

Proof. We first prove (a). Fix indices $j \in \{0, \dots, \min\{|X_0|, |X_2|\}\}$ and $k \in \{0, \dots, \min\{|X_0|, |X_4|\}\}$, and let $A_{j,k} \subseteq V(G)$ be any (j,k)-good set. We obtain a simplicial elimination ordering of $G[A_{j,k}]$ by first listing all vertices of $(X_0 \cup X_2 \cup X_5) \cap A_{j,k}$ (in any order), and then listing all vertices of $(X_3 \cup X_4) \cap A_{j,k}$ (in any order). So, by Theorem 3.2, $G[A_{j,k}]$ is chordal. Theorem 3.1 now guarantees that $G[A_{j,k}]$ is perfect, and consequently, $\chi(G[A_{j,k}]) = \omega(G[A_{j,k}])$. The fact that $\omega(G[A_{j,k}]) = |X_0|$ is immediate from the definition of a (j,k)-good set. Finally, by Lemma 7.3, $G \setminus X_0$ is $(4K_1, C_4, C_6, C_7, \text{5-pyramid})$-free; since $G \setminus A_{j,k}$ is an induced subgraph of $G \setminus X_0$, it follows that $G \setminus A_{j,k}$ is also $(4K_1, C_4, C_6, C_7, \text{5-pyramid})$-free. This proves (a).

It remains to prove (b). To simplify notation, we set $\chi := \chi(G)$. Let $\{S_1, \dots, S_\chi\}$ be a partition of $V(G)$ into stable sets.[78] Since X_0 is a clique, we see that exactly $|X_0|$ of the sets $S_1, \dots, S_\chi$ intersect X_0; by symmetry, we may assume that $S_1, \dots, S_{|X_0|}$ all intersect X_0,[79] and $S_{|X_0|+1}, \dots, S_\chi$ do not intersect X_0. Set $A := S_1 \cup \dots \cup S_{|X_0|}$. Since X_0 is complete to $X_1 \cup X_6 \cup Y \cup Z \cup W$ and anticomplete to $X_2 \cup X_3 \cup X_4 \cup X_5$, we have that $X_0 \subseteq A \subseteq X_0 \cup X_2 \cup X_3 \cup X_4 \cup X_5$. For each $i \in \{2,3,4,5\}$, set $X_i' := X_i \cap A$. Then $A = X_0 \cup X_2' \cup X_3' \cup X_4' \cup X_5'$. Furthermore, $\{S_1, \dots, S_{|X_0|}\}$ is a partition of $G[A]$ into stable sets, and $\{S_{|X_0|+1}, \dots, S_\chi\}$ is a partition of $G \setminus A$ into stable sets. Since $S_1, \dots, S_\chi$ are the color classes of an optimal coloring of G, we see that $\chi(G[A]) = |X_0|$ and $\chi(G \setminus A) = \chi - |X_0|$. Moreover, it is clear that $\omega(G[A]) = |X_0|$.[80] Now, let $j := |X_2'|$ and $k := |X_4'|$. Since $X_2' \cup X_3', X_3' \cup X_4', X_4' \cup X_5'$ are all cliques of $G[A]$, and are therefore of size at most $\omega(G[A]) = |X_0|$, we see that all the following hold: $0 \leq j \leq \min\{|X_0|, |X_2|\}$, $k \leq \min\{|X_0|, |X_4|\}$, $|X_3'| \leq \min\{|X_3|, |X_0| - j, |X_0| - k\}$, and $|X_5'| \leq \min\{|X_5|, |X_0| - k\}$.

Now, fix any (j,k)-good set $A_{j,k}$. It suffices to show that $\chi \geq \chi(G[A_{j,k}]) + \chi(G \setminus A_{j,k})$, for the reverse inequality trivially holds. First, we observe that $G \setminus A_{j,k}$ is isomorphic to an induced subgraph of $G \setminus A$; consequently, $\chi(G \setminus A_{j,k}) \leq \chi(G \setminus A) = \chi - |X_0|$, and it follows that $\chi \geq |X_0| + \chi(G \setminus A_{j,k})$. But by (a), we have that $\chi(G[A_{j,k}]) = |X_0|$, and so $\chi \geq \chi(G[A_{j,k}]) + \chi(G \setminus A_{j,k})$. This proves (b). □

Theorem 7.8. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph* G*;*
- ***Output:*** *One of the following:*
  - *An optimal coloring of* G*,*
  - *The true statement that either* G *is not* $(4K_1, C_4, C_6)$*-free or* G *does not contain an induced* C_7*;*
- ***Running time:*** $O(n^3)$.

[78] So, $S_1, \dots, S_\chi$ are the color classes of some optimal coloring of G.

[79] Note that $|S_i \cap X_0| = 1$ for all $i \in \{1, \dots, |X_0|\}$.

[80] Indeed, X_0 is a clique of $G[A]$, and so $\omega(G[A]) \geq |X_0|$. On the other hand, $\omega(G[A]) \leq \chi(G[A]) = |X_0|$.

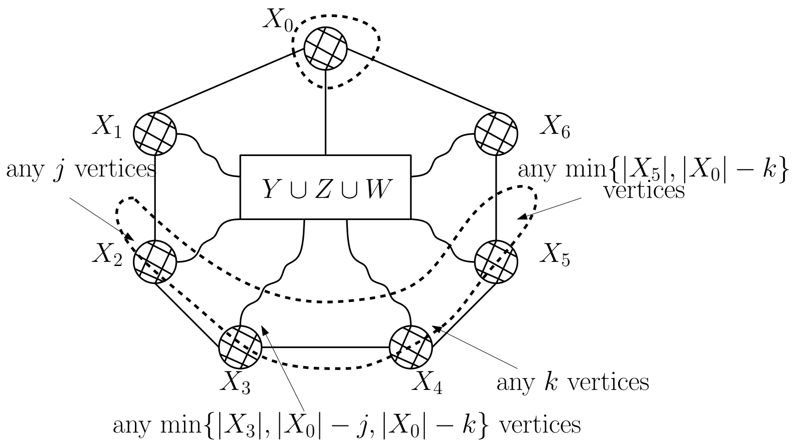


Figure 7.1: A graph G with a special partition $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$. A (j, k)-good set is the union of sets represented by the two dashed bags.

Proof. We first call the $O(n^2)$ time algorithm from Theorem 7.1 with input G. If the algorithm returns the answer that either G is not $(4K_1, C_4, C_6)$-free or G does not contain an induced C_7, the we return this answer as well, and we stop. From now on, we assume that the algorithm returned the answer that G is $(4K_1, C_4, C_6)$-free and contains an induced C_7, together with a special partition $(X_0, \dots, X_6; Y_0, \dots, Y_6; Z_0, \dots, Z_6; W)$ of G. Set $X := \bigcup_{i \in \mathbb{Z}_7} X_i$, $Y := \bigcup_{i \in \mathbb{Z}_7} Y_i$, and $Z := \bigcup_{i \in \mathbb{Z}_7} Z_i$.

By Lemma 7.3(c), there exists an index $i \in \mathbb{Z}_7$ such that X_i is complete to $Y \cup Z \cup W$; clearly, such an index i can be found in $O(n^2)$ time, and by symmetry, we may assume that $i = 0$. By Lemma 7.3, the graph $G \setminus X_0$ is $(4K_1, C_4, C_6, C_7, \text{5-pyramid})$-free.

Now, let F be the 22-vertex graph with vertex set $V(F) = \{x_0, \dots, x_6\} \cup \{y_0, \dots, y_6\} \cup \{z_0, \dots, z_6\} \cup \{w_0\}$ (with indices in $\mathbb{Z}_7$), and with adjacency as follows:

- $x_0, \dots, x_6, x_0$ is a 7-hole in F;
- $\{y_0, \dots, y_6\}$ and $\{z_0, \dots, z_6\}$ are cliques of F;
- for all $i \in \mathbb{Z}_7$, x_i is complete to $\{y_i, y_{i+3}, y_{i+6}, z_i, z_{i+3}, z_{i+4}, z_{i+5}, z_{i+6}\}$ and anticomplete to $\{y_{i+1}, y_{i+2}, y_{i+4}, y_{i+5}, z_{i+1}, z_{i+2}\}$;
- for all $i \in \mathbb{Z}_7$, y_i is complete to $\{z_0, \dots, z_6\} \setminus \{z_{i+2}\}$ and nonadjacent to z_{i+2};
- w_0 is complete to $V(F) \setminus \{w_0\}$.

Further, we define the weight function $w : V(F) \to \mathbb{N}$ as follows. We set $w(w_0) := |W|$, and for all $i \in \mathbb{Z}_7$, we set $w(x_i) := |X_i|$, $w(y_i) := |Y_i|$, and $w(z_i) := |Z_i|$. Clearly, the weight function w can be computed in $O(n)$ time. We note that F^w is isomorphic to G.

Next, we compute the set $\mathcal{C}$ of all cliques of F, as well as the set $\mathscr{F}(F)$; since $|V(F)| = 22$, this can be done in $O(1)$ time. For all indices $j \in \{0, \dots, \min\{|X_0|, |X_2|\}\}$ and $k \in \{0, \dots, \min\{|X_0|, |X_4|\}\}$, we define the weight function $w_{j,k} : V(F) \to \mathbb{N}$ and the number $\chi_{j,k}$ as follows:

- for all $v \in V(F) \setminus \{x_0, x_2, x_3, x_4, x_5\}$, $w_{j,k}(v) := w(v)$;
- $w_{j,k}(x_0) := 0$;
- $w_{j,k}(x_2) := w(x_2) - j$;
- $w_{j,k}(x_3) := w(x_3) - \min\{w(x_3), w(x_0) - j, w(x_0) - k\}$;
- $w_{j,k}(x_4) := w(x_4) - k$;
- $w_{j,k}(x_5) := w(x_5) - \min\{w(x_5), w(x_0) - k\}$;
- $\chi_{j,k} := \max\Big(\{\omega(F, w_{j,k})\} \cup \{\lceil w_{j,k}(A)/2 \rceil + w_{j,k}(B) \mid (A, B) \in \mathscr{F}(F)\}\Big)$.

Since $|V(F)| = 22$, we see that families $\{w_{j,k}\}$ and $\{\chi_{j,k}\}$ can be computed in $O(n^2)$ time.

We now find indices j, k for which $\chi_{j,k}$ is minimum, we let $A_{j,k}$ be any (j, k)-good set (this is defined as in Lemma 7.7), and for those indices j, k, we form graphs $G[A_{j,k}]$ and $G \setminus A_{j,k}$; this can be done in further $O(n^2)$ time.

Claim 1. $\chi(G) = \chi(G[A_{j,k}]) + \chi(G \setminus A_{j,k})$.

Proof of Claim 1. By construction, $F^{w_{j,k}}$ is isomorphic to $G \setminus A_{j,k}$, and by Lemma 7.7(a), $G \setminus A_{j,k}$ is $(4K_1, C_4, C_6, C_7, 5\text{-pyramid})$-free. Consequently, $F^{w_{j,k}}$ is $(4K_1, C_4, C_6, C_7, 5\text{-pyramid})$-free, and so Lemma 7.6 guarantees that $\chi(F, w_{j,k}) = \chi_{j,k}$. Thus, $\chi(G \setminus A_{j,k}) = \chi(F^{w_{j,k}}) = \chi(F, w_{j,k}) = \chi_{j,k}$.

By Lemma 7.7(b), there exist $j' \in \{0, \dots, \min\{|X_0|, |X_2|\}\}$ and $k' \in \{0, \dots, \min\{|X_0|, |X_4|\}\}$ such that $\chi(G) = \chi(G[A_{j',k'}]) + \chi(G \setminus A_{j',k'})$. Analogously to the above, we have that $\chi(G \setminus A_{j',k'}) = \chi_{j',k'}$, and so by the choice of j, k, we have that $\chi(G \setminus A_{j,k}) = \chi_{j,k} \leq \chi_{j',k'} = \chi(G \setminus A_{j',k'})$. On the other hand, Lemma 7.7(a) guarantees that $\chi(G[A_{j,k}]) = |X_0| = \chi(G[A_{j',k'}])$. We now have that $\chi(G[A_{j,k}]) + \chi(G \setminus A_{j,k}) \leq \chi(G[A_{j',k'}]) + \chi(G \setminus A_{j',k'}) = \chi(G) \leq \chi(G[A_{j,k}]) + \chi(G \setminus A_{j,k})$, and it follows that $\chi(G) = \chi(G[A_{j,k}]) + \chi(G \setminus A_{j,k})$. This proves Claim 1. ♦

We now complete the description of the algorithm. First, we obtain a simplicial elimination ordering of $G[A_{j,k}]$ by first listing all vertices of $(X_0 \cup X_2 \cup X_5) \cap A_{j,k}$ (in any order), and then listing all vertices of $(X_3 \cup X_4) \cap A_{j,k}$ (in any order). We then obtain an optimal coloring of $G[A_{j,k}]$ by coloring the vertices of $G[A_{j,k}]$ greedily, using the reverse of this simplicial elimination ordering; this takes $O(n^2)$ time. Next, by Lemma 7.7(a), the graph $G \setminus A_{j,k}$ is $(4K_1, C_4, C_6, C_7, 5\text{-pyramid})$-free; so, we can obtain an optimal coloring of $G \setminus A_{j,k}$ by calling the $O(n^3)$ time algorithm from Lemma 6.8. After possibly renaming colors, we may assume that our colorings of $G[A_{j,k}]$ and $G \setminus A_{j,k}$ use disjoint color sets. We now take the union of these two colorings, and we obtain a proper coloring of G; by Claim 1, this coloring of G is in fact optimal. We now return our coloring of G, and we stop.

Clearly, the algorithm is correct, and its running time is $O(n^3)$. □

8 The main results

Theorem 8.1. *Every $(4K_1, C_4, C_6)$-free graph has at most $|V(G)|$ maximal cliques.*

Proof. This follows from Corollary 5.2 and Theorem 7.4. □

Theorem 8.2. *There exists an algorithm with the following specifications:*

- ***Input:*** *A graph G;*
- ***Output:*** *One of the following:*
  - *The true statement that G is $(4K_1, C_4, C_6)$-free, together with the list of all maximal cliques of G, an optimal coloring of G, and a minimum clique cover of G;*
  - *The true statement that either G is not $(4K_1, C_4, C_6)$-free;*
- ***Running time:*** *$O(n^3)$.*

Proof. This follows immediately from Theorems 4.4, 5.1, 5.4, 6.8, 7.1, and 7.8. □

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